Extensions of Lamzouri's method to fixed Dedekind zeta functions

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September 5, 2026

Abstract

We extend Lamzouri's Hilbert-space distinct-zero certificate to every integer multiplicity threshold. An unconditional strict-support pair formula, proved directly with the complex zeros retained, then gives distinct-zero proportions for every fixed number field. The arithmetic parameter is the permutation rank, equivalently the double-coset count for the action on its embeddings. In degrees two, three, and four the degree-only lower bounds are respectively 0.4449187130, 0.3057123676, and 0.2334036149. For fixed abelian fields the factorwise argument also gives 0.6725007036 of zeros on the critical line, counting multiplicity. No assertion of product simplicity is inferred from this latter result. A projection argument retaining the geometry of the height window gives an additional explicit, strictly positive improvement of the distinct bounds, exceeding 10^{-15} in the quadratic case.

Keywords. Dedekind zeta function; distinct zeros; pair correlation; multiplicity; Hilbert-space method; Chebotarev theorem.

1 Introduction

1.1 Overview and motivation

The zeros of a Dedekind zeta function encode arithmetic information about a number field, but their multiplicities introduce a difficulty that is already visible in the simplest extensions of $\mathbb{Q}$. For a quadratic field one has $\zeta_K(s) = \zeta(s)L(s, \chi_K)$, where χ_K is the nonprincipal primitive quadratic character associated with K. A common nontrivial zero of the two factors is necessarily a multiple zero of the product. More generally, the factorization of a Dedekind zeta function reflects the representation theory of its normal closure. Consequently, a statement about simple zeros of individual factors does not automatically become a statement about simple zeros of the Dedekind zeta function.

This paper asks how much distinctness can be proved without first assuming that the zeros lie on the critical line. Throughout, the field $K/\mathbb{Q}$ is fixed, $n = [K : \mathbb{Q}]$, and T tends to infinity. All three counts below concern nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \beta < 1$ and $0 < \gamma \le T$. We distinguish the total count $N_K(T)$, with analytic multiplicity, from the count $D_K(T)$ of distinct complex zero locations and from the count $N_{K,0}(T)$ on $\beta = 1/2$, with multiplicity. A simple zero has analytic multiplicity one. Every percentage in this introduction is normalized by $N_K(T)$, unless a different zeta function is specified. These statistics answer different questions: GRH would locate all nontrivial zeros on the line, but would not by itself make them simple. In particular, none of our distinct-zero percentages is asserted to be a simple-critical-line percentage.

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Our starting point is Lamzouri's Hilbert-space argument [1], which converts unconditional pair-correlation estimates into zero-count bounds. Its central advantage is that a global quadratic energy remains nonnegative even when individual kernel evaluations at complex zero differences have no positivity property. For Dedekind zeta functions the natural energy scale grows with the degree, so varying the multiplicity threshold improves the resulting distinct-zero bound. We develop an integer-threshold version, prove the required fixed-field analytic input, and retain additional information about the length of the zero window to obtain, for $n \geq 2$, a strictly positive geometric refinement.

1.2 Past work and related literature

There are two complementary traditions behind zero-proportion results. The mollifier method obtains information from mean values of a zeta or L-function and its derivatives. Levinson's work [2] and Conrey's refinement [3] are foundational examples for ζ; the long-mollifier work of Pratt, Robles, Zaharescu and Zeindler [4] develops this direction further. Distinctness can also be extracted from derivative-zero information, as in Farmer [5] and Ki–Lee [6]. These results supply useful unconditional benchmarks, but they do not remove the need to control multiplicity in a product of different L-functions.

The second tradition starts from Montgomery's pair-correlation theorem [7]. Under RH, positivity on real zero differences turns its short-support formula into a simple-zero proportion. The Montgomery–Taylor optimization, recorded in Montgomery [8], improves the underlying test function. Chirre, Gonçalves and de Laat [9] use semidefinite programming to obtain stronger RH-dependent bounds. The relevance of these works here is the quadratic-energy viewpoint, not a license to use their positivity arguments away from the critical line.

Aryan [10] and Baluyot, Goldston, Suriajaya and Turnage-Butterbaugh [11] established forms of the analytic pair-correlation calculation that keep the actual complex zeros. The later paper [12, Sections 2–3] corrects the pointwise error term in [11], while explicitly preserving its integrated applications, including Lemma 5. Our fixed-field error estimates are proved directly in Section 4. The remaining obstacle is algebraic: an entire kernel cannot generally be treated as termwise nonnegative on all such differences. The 2026 work of Alpöge and Furman [14], whose authors credit Claude with discovering and writing the argument and report verifying it, uses a finite compression of Weil's Hermitian form, together with rank and inertia estimates, to overcome this obstacle. Lamzouri [1] gives a different proof based on a finite conjugation-invariant multiset in a Hilbert space. Both recover the same optimised constants, approximately 67.25007% for simple critical-line zeros of ζ and 83.62504% for distinct zeros. Our finite argument extends the latter Hilbert-space mechanism to every integer multiplicity threshold.

For Dedekind zeta functions, Conrey, Ghosh and Gonek [15, 16] studied simple zeros in quadratic extensions. They obtained an unconditional power-sized lower count, and a simple-zero proportion $1/54$ assuming RH for ζ; the latter improves to $1/27$ assuming GRH for the quadratic ζ_K [16, Theorem 1 and the following Remark]. Wu and Zhao [17] subsequently improved the unconditional lower count in the quadratic setting. Such counting results and a fixed positive percentage are different conclusions. De Laat, Rolen, Tripp and Wagner [18] prove a pair-correlation theorem for fixed abelian extensions under GRH and obtain explicit distinct-zero bounds by semidefinite programming. Their stated quadratic bound is 45.85%; our unconditional quadratic scalar bound, $44.4918713\ldots\%$, is numerically smaller. Moreover, their moment estimates imply stronger cubic and quartic consequences than their printed distinct-zero corollaries. Section 8 makes these comparisons with the hypotheses and denominators kept fixed.

The earlier Dedekind literature is not confined to abelian fields. Alsharif [19, Theorem 1.7 and Corollary 1.8, pp. 9–10] gives, under GRH, a pair formula for Galois fields of degree n and the distinct-zero bound $3/(3n+1)$. The seven-author project *Pair correlation for Dedekind zeta functions* [21] has public theorem statements in Tripp's 2022 dissertation [20, Chapter VI, pp. 114–117], where they are attributed to Alsharif, Gibson, de Laat, Milinovich, Rolen, Tripp and Wagner, with the date 2020. Under GRH, their Theorems VI.2.1–VI.2.2 give a pair-correlation upper bound and squared-multiplicity estimates for *arbitrary* number fields; Corollary VI.2.3 gives a positive distinct-zero proportion for every degree. Thus neither treating nonabelian fields nor obtaining an arbitrary-field positive proportion under GRH is new here. Alsharif's proof also identifies earlier analytic inputs from Murty–Perelli [22] and a Galois prime-coefficient calculation from Milinovich–Turnage-Butterbaugh [23]. Section 8.2 compares the original statements, their normalizations, and their implications in detail.

The factorwise Dirichlet theory is relevant but must be used with care. Garunkštis and Paliulionytė [24] obtain a Dirichlet analogue of the unconditional pair-correlation framework, with applications to simplicity under additional hypotheses; Alpöge–Furman [14, Theorem B] also give the fixed primitive Dirichlet extension of their 2026 result. An abelian Dedekind zeta function is a product of fixed primitive Dirichlet factors. Their critical-line multiplicities add exactly, but their simple zeros can coincide. Family-averaged large-sieve results, such as Chandee, Lee, Liu and Radziwiłł [25] under GRH, concern a different limiting regime and are not substituted for fixed-field estimates in this paper. The work of Hu, Kaneko, Martin and Schildkraut [26], including its use of Stark's local Artin-holomorphy theorem [27], further illustrates why representation-theoretically forced multiplicities must be respected.

Finally, the variational constant used below is not new. Das, Ismoilov and Ramos [28, Theorem 5 and Corollary 9] place the generalised Montgomery–Taylor optimization in a broader Fourier-analytic framework. We give a direct proof of the needed specialization and use it with a complex-zero certificate. The analytic arithmetic inputs are likewise classical: a fixed-extension effective Chebotarev theorem [29, 30] identifies the prime-coefficient second moment, and the Montgomery–Vaughan mean-value inequality [31] controls the Dirichlet-polynomial error. Neither input requires global Artin holomorphy in the application made here.

1.3 What is novel: principal contributions claimed here

The principal contributions claimed in this paper are the unconditional arbitrary-field analytic theorem, its rank-sensitive distinct-zero consequence, and, for $n \geq 2$, its geometric refinement. These are claims about the statements and combinations proved in this paper, not claims to have originated their Hilbert-space, explicit-formula, or variational ingredients. The comparisons below specify the relevant earlier results; the following subsection identifies what is derived from them.

1. An unconditional fixed-field Dedekind pair formula. We prove the strict-support pair formula directly for the actual complex zeros of ζ_K, for every fixed number field K. Let M be the normal closure of $K/\mathbb{Q}$, $G = \mathrm{Gal}(M/\mathbb{Q})$, and $H = \mathrm{Gal}(M/K)$. Its arithmetic parameter is

$$r = |H\backslash G/H|,$$

the number of H-orbits on G/H, or equivalently the permutation rank of the action of G on the embeddings of K into M. Thus $1 \leq r \leq n$, and $r = n$ when $K/\mathbb{Q}$ is Galois. Write $P(g) = \#\,\mathrm{Fix}_{G/H}(g)$ for the permutation character. The coefficient follows from Chebotarev and the character identity $|G|^{-1}\sum_{g\in G} P(g)^2 = r$. The character identity and Chebotarev input are classical; the claimed contribution is their implementation in the complex-zero formula with the fixed-field error estimates of Lemma 4. For a precise comparison, De Laat, Rolen, Tripp and Wagner [18, Theorem 1.1] assume GRH and an abelian extension in their pair-correlation theorem itself. Alsharif [19, Theorem 1.7] already gives the Galois case under GRH, and the seven-author theorem in [20, Theorem VI.2.1] states an upper bound for arbitrary fields under GRH. The Dedekind application of Das, Ismoilov and Ramos [28, Corollary 9] likewise assumes GRH and is concerned with averages of the form factor. The distinction from these public statements is the unconditional complex-zero formula with its exact general permutation-rank coefficient, not the degree range or a larger Fourier support obtained by changing scale. The pair formula retains the real parts of the zeros. In its counting applications, the Hilbert certificate replaces the termwise positivity that GRH would provide; it is not an input to the analytic pair formula itself. No abelian hypothesis or global Artin-holomorphy conjecture is used. The price is that the field is fixed: our error terms are not uniform in the degree or discriminant. The complex-zero analytic framework is inherited from [10, 11, 12]; we do not present retaining the real parts of the zeros as a new idea.

2. Unconditional rank-sensitive distinct-zero proportions. The trigonometric variational constant itself is known from generalized Montgomery–Taylor theory [28]; we do not claim it as new. Our claimed contribution, Theorem 1, is to combine that optimizer with the all-threshold complex-zero certificate and the fixed-field arithmetic coefficient. If $n = [K : \mathbb{Q}]$, this gives

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq b_{n,r} := \frac{2n+1-C_{n,r}}{n(n+1)}, \qquad C_{n,r} = \frac{r}{2n} + \sqrt{\frac{r}{2}}\cot\left(\frac{\sqrt{r/2}}{n}\right).$$

The scalar constant $C_{n,r}$ is strictly increasing in r at fixed n, as follows from the variational formula in Section 5. Thus $r \leq n$ gives a degree-only lower bound $b_{n,n}$; a field with $r < n$ has a strictly stronger bound. For $r = n$, the bounds in degrees two, three and four are respectively $44.4918713\ldots\%$, $30.5712367\ldots\%$, and $23.3403614\ldots\%$.

For comparison, the 2019 paper [18, Corollary 1.3] prints, under GRH and for abelian fields, 45.85%, 27.94%, and 11.27% in these degrees. Its stronger squared-multiplicity estimates, when inserted into the elementary integer inequality (53), imply $44.6233333\ldots\%$, $30.6400000\ldots\%$, and $23.3825000\ldots\%$ under GRH; the additional quadratic simple-zero input from [16] raises the first figure to $45.85790123\ldots\%$. The same squared-multiplicity estimates are stated for arbitrary fields in [20, Theorem VI.2.2], so these GRH comparisons are not restricted to abelian fields. In particular, the reciprocal bounds stated in [20, Corollary VI.2.3] evaluate in degrees three and four to $30.0914780\ldots\%$ and $23.1294090\ldots\%$, rather than the smaller figures printed in [18]. Our degree-only theorem improves both sets of printed cubic and quartic corollaries while removing GRH, but it is not numerically stronger than their integer-moment consequences under GRH.

There is also a direct unconditional quadratic comparison. Since $L(s, \chi_K)$ is entire, the quadratic factorization gives $D_K(T) \geq D_\zeta(T)$, even at common zeros of the two factors. Together with $N_K(T) \sim 2N_\zeta(T)$, this shows that Lamzouri's $83.62503518\ldots\%$ distinct-zero theorem for ζ already implies $41.81251759\ldots\%$ for a fixed quadratic Dedekind zeta function. Our scalar bound improves this inherited consequence by $2.67935371\ldots$ percentage points. This comparison is a consequence of the cited Riemann-zeta result, not a claim about the complete history of unconditional Dedekind bounds.

3. A strict improvement beyond the optimized scalar moment. For $n \geq 2$, Theorem 13 gives a strict improvement over $b_{n,r}$. The scalar bound uses only total multiplicity, the distinct count, and one aggregate kernel energy. The normalized zero window contains additional usable information. For the optimized cosine kernel and any fixed diameter bound, the sum of the three squared kernel values between three real locations has a positive lower bound, uniform over their locations; the bound depends on the kernel and the diameter. Projecting away the span generated by nonreal conjugate pairs preserves this three-point information up to a controlled loss, after which interval packing produces

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq b_{n,r} + \epsilon_{n,r}, \qquad \epsilon_{n,r} > 0.$$

The additional contribution claimed in Theorem 13 is this combination of projection, the three-point kernel gap, and packing at the actual zero-height density. In contrast to the aggregate Hilbert-space and rank–inertia estimates in [1, 14], it uses the locations of the real points in the height window and the nonreal defect $\Delta_n \geq N_C$, defined below. The explicit increment is extremely small—more than 10^{-15} in the fraction D_K/N_K in the quadratic case—so its importance is structural rather than numerical. The exact model in Section 8 shows sharpness for a specified flat kernel, even with its full factor-labelled second-moment matrix. It is not an equality example for the optimized cosine kernel and does not exclude improvements from other kernels or additional moment information.

1.4 What is derived: inherited techniques and corollaries

1. The threshold certificate extends an existing mechanism. Lamzouri's [1, Proposition 2.1] Hilbert-space representation, conjugate-pair decomposition, nested subspaces, and quadratic tangent inequalities are the starting point of Lemma 3. We extend the multiplicity levels as follows. Let Z be a finite nonempty multiset of complex numbers, invariant under conjugation with multiplicities. Take a real even compactly supported $\eta \in L^2(\mathbb{R})$ with $\int_{\mathbb{R}} \eta(u)^2\,du = 1$, and define

$$K_\eta(z) = \int_{\mathbb{R}} \eta(u)^2 e^{-2\pi i z u}\,du, \qquad E = \sum_{z,w\in Z} K_\eta(z-w)^2,$$

where the double sum includes multiplicities and the square is an ordinary square, not an absolute square. Let N be total multiplicity, D the number of distinct locations, S the number of simple real locations, and N_C the total multiplicity of nonreal locations. Then, for every integer $k \geq 1$,

$$D \geq \frac{(2k+1)N - E}{k(k+1)} + \frac{k-1}{k+1}S, \qquad \Delta_k := E - (2k+1)N + k(k+1)D \geq N_C.$$

At $k = 1$, the displayed distinctness bound is Lamzouri's $D \geq (3N - E)/2$; his separate $S \geq 2N - E$ inequality is also retained. The rank–trace proof in [14, Lemma 3.2 and Section 7.2(c)] provides a closely related quadratic-tangent and integrality mechanism. Thus the extension lies in the all-integer formulation and the explicitly retained correction and defect terms, not in a new Hilbert-space filtration principle. These statements still require an operator argument: one cannot apply $(m-k)(m-k-1) \geq 0$ termwise to complex kernel evaluations. Moreover, the elementary rank–trace estimate already gives $D \geq N^2/E$ for the representing operator, whose trace is N, rank is at most D, and squared Hilbert–Schmidt norm is E. Since $1 \leq D \leq N$, this also gives $E/N \geq 1$. Choosing $k = \lfloor E/N \rfloor$ sharpens this positive bound when E/N is nonintegral and matches it when E/N is integral. Thus integer thresholds are not what first makes a positive distinct-zero proportion possible.

2. The optimizer, weight removal, and arithmetic inputs are inherited. The constant $C_{n,r}$ is the reciprocal of the reproducing-kernel value in [28, Theorem 5] with $c_1 = n$, $c_2 = r/n$, and $\Delta = 1$. Its hypothesis holds because $(c_2/c_1)\Delta^2 = r/n^2 \leq 1 < 5/3$; the cosine optimization belongs to generalized Montgomery–Taylor theory [8, 28]. Section 5 supplies a direct proof for use here, not a new extremal constant. The differential correction removing the rational pair weight follows [1, Section 3]. The Frobenius fixed-point description, character orthogonality, fixed-extension Chebotarev estimates [29, 30], and Montgomery–Vaughan mean-value inequality [31] are likewise established inputs, not separate novelty claims. In the Galois case, the prime-coefficient second moment already occurs in [19, Lemma 5.1], which invokes [23, Lemma 5.2]. Alsharif's explicit-formula input [19, Proposition 4.1] is attributed to [22, equation (30)]. These precedents are distinct from our fixed-field control of the complex-zero errors.

3. The endpoint corollary is a quantitative fixed-field adaptation. Corollary 12 allows a real even $f \in C_c^2(\mathbb{R})$ supported in $[-1,1]$. After division of the pair sum by $N_K(T)$, its error is $O_K(\|f\|_{C^2}/\log T)$, where $\|f\|_{C^2} := \sum_{j=0}^{2} \|f^{(j)}\|_\infty$. For the Riemann zeta function, [11, Lemma 5] already admits endpoint support for real even integrable tests that are Lipschitz at zero; this application remains valid after the correction in [12, Section 2]. Accordingly, the contribution here is the all-field implementation and stated quantitative estimate, not the endpoint principle itself. The proof uses the boundary conditions $f(\pm 1) = f'(\pm 1) = 0$ and the resulting quadratic vanishing and does not assert this error bound for the nonsmooth optimizer.

4. The abelian critical-line percentage is a derived corollary. For a fixed abelian field, the primitive Dirichlet theorem in [14, Theorems A–B], applied factor by factor, gives

$$\liminf_{T\to\infty} \frac{N_{K,0}(T)}{N_K(T)} \geq c_0 := \frac{3}{2} - \frac{1}{\sqrt{2}}\cot(1/\sqrt{2}) = 0.6725007036\ldots,$$

with multiplicity. This is a factorwise consequence, not an independent new zero-proportion theorem; the argument is summarized below. Nor does it imply that this proportion of zeros is simple in the Dedekind product, because zeros of different factors may coincide. The principal advance of the paper is instead the unconditional distinct-zero theorem for arbitrary fixed fields and its geometric refinement. No GRH, conjectural pair correlation, or global Artin-holomorphy assumption is made in these lower bounds, and no uniformity as the field varies is claimed.

Scope of the priority comparison. The GRH-dependent numerical consequences of [19, 18, 20, 16] and the unconditional quadratic consequence of [1] displayed above are derived comparisons, not additional independent theorems of this paper. The comparison with [19, 20] is based on Alsharif's full thesis and the public statements of the seven-author results in Tripp's dissertation. Those statements establish the GRH-dependent precedents described above; neither source contains the unconditional arbitrary-field theorem or the complex-zero geometric refinement asserted here. Chapter VI of [20] is a summary of results, not a full proof of the seven-author project. The project remains listed as in preparation in [21] as accessed September 7, 2026. Our priority comparison is therefore with its publicly documented statements, not with any unavailable later version. The remaining uncertainty concerns possible later versions, not whether arbitrary-field GRH results had already been stated. This bounded comparison does not certify originality across the entire literature or replace independent verification of our proofs.

1.5 Overview of the proof strategy

1. A multiplicity-sensitive Hilbert certificate. We use the normalization

$$z_\rho = i(\rho - 1/2)\log T/(2\pi), \qquad 0 < \gamma \leq T.$$

The functional equation together with conjugation of ζ_K gives the involution $\rho \mapsto 1 - \bar{\rho}$, which preserves the height window and multiplicity. Since $z_{1-\bar{\rho}} = \overline{z_\rho}$, the resulting multiset is conjugation-invariant; critical-line zeros correspond exactly to real points. Use K_η and E as defined above. A real Hilbert-space representation makes E the squared Hilbert–Schmidt norm of a symmetric finite-rank operator, although individual summands need not be real or nonnegative. Nested subspaces organised by multiplicity yield, for every integer $k \geq 1$,

$$D \geq \frac{(2k+1)N - E}{k(k+1)}.$$

The proof also retains $\Delta_k \geq N_C$, counting every nonreal point with its multiplicity. This extra information is needed later; a scalar rank–trace bound alone would lose it.

2. An unconditional arithmetic evaluation. We retain the real parts of the zeros in a rational-kernel explicit formula, justified directly by contour integration, and in the norm identity. Local zero counts control the two ends of the sharp window $0 < \gamma \leq T$. At a rational prime p unramified in M, the coefficient of p^{-s} in $-\zeta_K'/\zeta_K(s)$ is $P(\mathrm{Frob}_p)\log p$. Chebotarev and character orthogonality identify the corresponding second-moment coefficient as $r = |H\backslash G/H|$. This is a statement about a permutation character; it does not require holomorphy of its separate Artin factors. For the zero-proportion theorems, the resulting pair formula is used only for fixed smooth tests supported strictly inside $(-1,1)$ at the $\log T$ scale. The globally smooth endpoint corollary is separate.

3. Weight removal and optimization. Now choose $\eta \in C_c^\infty((-1/2,1/2))$ with the same reality, evenness, and normalization. Put $p = \eta^2$ and $q = p * p$. The differential correction $q - q''/(4\log^2 T)$ removes the rational weight exactly, as shown in (43). Although its coefficient depends on T, the pair formula is applied separately to the two fixed tests q and q''. The unweighted energy satisfies $E/N_K(T) = J_{n,r}(p) + o(1)$, where

$$J_{n,r}(p) = n\int_{-1/2}^{1/2} p(x)^2\,dx + \frac{r}{n}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |x-y|p(x)p(y)\,dx\,dy.$$

Among real $L^2([-1/2,1/2])$ densities of integral one, its unique minimiser is the positive cosine density displayed in Section 5. Its value is $C_{n,r}$, with $n < C_{n,r} < n + r/(3n) \leq n + 1/3$. Choosing $k = n$ gives the main scalar bound. Smooth cutoffs approximate the optimiser: each cutoff is fixed before $T \to \infty$, and only then is the cutoff removed. Thus neither an endpoint pair theorem nor uniformity for a moving test function is assumed.

4. A strict gain from the zero-window geometry. Here $n \geq 2$. The real coordinates of the normalised zeros lie in $[-H_0, 0)$, where $H_0 = T\log T/(2\pi)$ and $H_0/N_K(T) \to 1/n$. For the cosine kernel, an explicit trigonometric identity forces a positive total squared correlation among every three real locations within a fixed diameter. Its uniform approximation by the smooth cutoff kernels preserves a positive three-point gap. For each fixed cutoff, we project away the real span of the vectors g_z, h_z associated with nonreal conjugate pairs in Lemma 3. The total loss in the squared norms of the real vectors, weighted by their multiplicities, satisfies $\mathcal{L} \leq \sqrt{N_C E} - N_C$ by (49). Disjoint triples have interaction graphs of degree two, which controls the accumulated projection loss. Packing these triples into the height window and combining with the earlier nonreal defect reduces the remaining estimate to a lower bound from $\min_{x\geq 0}\max\{x, a - B\sqrt{x}\} > 0$, where x represents N_C/N and $a, B > 0$ are specified in Theorem 13. First $T \to \infty$ at each fixed cutoff, and then the cutoff is removed. This gives $\epsilon_{n,r} > 0$ without treating complex cross terms as positive or applying the pair formula to a nonsmooth kernel.

5. A separate abelian critical-line argument. For an abelian field, write $\zeta_K = \prod_{\chi\in X_K} L(s,\chi)$ with n primitive Dirichlet factors, including the factor ζ. They are holomorphic in the open critical strip, so their critical-line multiplicities add exactly, including at common zeros. Each fixed factor satisfies the simple-critical-line bound c_0 of [14, Theorems A–B]; its multiplicity-weighted critical-line count is at least its simple count. Summing over these finitely many factors gives the stated multiplicity-weighted fraction. Section 6 also explains the degree-one adaptation of the pair formula and Lamzouri's certificate. No bound on common zeros is used, and no simple-product conclusion follows.

1.6 Organization of the paper

Section 2 states the results and fixes the counting conventions. Section 3 proves the integer-threshold Hilbert certificate and its refinement. Section 4 establishes the unconditional fixed-field pair formula, including the arithmetic coefficient and boundary estimates. Section 5 removes the rational weight and proves the scalar extremal formula. Section 6 treats abelian critical-line zeros and explains the limitations imposed by product multiplicity. Section 7 proves the strict geometric improvement. Section 8 compares unconditional and GRH-dependent bounds, gives an exact finite sharpness model, and describes the remaining simplicity limitations.

2 Statements and scope

Let $K/\mathbb{Q}$ be a fixed number field of degree $n = [K : \mathbb{Q}] \geq 1$. For $T > 0$, let $\mathcal{Z}_K(T)$ be the finite set of distinct zeros $\rho = \beta + i\gamma$ of ζ_K with $0 < \beta < 1$ and $0 < \gamma \leq T$, and put $m_\rho = \operatorname{ord}_\rho \zeta_K \geq 1$. All four counting functions below use this same height window:

$$N_K(T) = \sum_{\rho \in \mathcal{Z}_K(T)} m_\rho, \qquad D_K(T) = |\mathcal{Z}_K(T)|,$$
$$N_{K,0}(T) = \sum_{\substack{\rho \in \mathcal{Z}_K(T) \\ \operatorname{Re}\rho = 1/2}} m_\rho, \quad N_K^{\text{simple}}(T) = |\{\rho \in \mathcal{Z}_K(T) : m_\rho = 1\}|.$$

Thus $D_K(T)$ counts complex locations, not just distinct ordinates; $N_{K,0}(T)$ includes multiplicity, whereas $N_K^{\text{simple}}(T)$ counts simple zeros anywhere in the strip. Real zeros ($\gamma = 0$) are excluded from all four counts. In particular, $N_K^{\text{simple}}(T) \leq D_K(T) \leq N_K(T)$ and $N_{K,0}(T) \leq N_K(T)$. The estimates for $D_K(T)$ below concern distinct locations in the whole strip, not a count restricted to simple or critical-line zeros.

Fix an embedding of K in its normal closure M over $\mathbb{Q}$, and set $G = \operatorname{Gal}(M/\mathbb{Q})$ and $H = \operatorname{Gal}(M/K)$. Then $[G : H] = n$. Define

$$r = |H\backslash G/H|, \qquad C_{n,r} = \frac{r}{2n} + \sqrt{\frac{r}{2}} \cot\left(\frac{1}{n}\sqrt{\frac{r}{2}}\right).$$

The coset space G/H identifies with the embeddings of K into M. Its H-orbits are the double cosets, so $1 \leq r \leq n$; if $n > 1$, the fixed coset H and its nonempty complement give $r \geq 2$. For a Galois extension $K/\mathbb{Q}$, one has $M = K$, $H = \{1\}$, and $r = n$. The cotangent is well-defined because $0 < \sqrt{r/2}/n \leq 1/\sqrt{2} < \pi/2$. All limits keep K fixed; no uniformity in the degree or discriminant is asserted. The normalization $N_K(T) = nT \log T/(2\pi) + O_K(T)$, proved in (27), ensures that the denominators below are positive for all sufficiently large T.

Theorem 1. *For every fixed number field $K/\mathbb{Q}$ as above, without assuming GRH or the Artin holomorphy conjecture, one has*

$$\boxed{\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq b_{n,r} := \frac{2n + 1 - C_{n,r}}{n(n+1)}.} \tag{1}$$

In particular,

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq \frac{1}{n} - \frac{r}{3n^2(n+1)} \geq \frac{3n+2}{3n(n+1)}. \tag{2}$$

The optimized bound (1) is strictly larger than the first bound in (2). For $n \geq 2$, Theorem 13 further increases (1) by an explicit $\epsilon_{n,r} > 0$.

Derivation from the results proved below. Section 5 applies the exact weight-removal identity (43) and Lemma 4 to fixed smooth cutoffs $p_j = \eta_j^2$ of the positive cosine minimizer. Their limiting normalized energies are $C_j = J_{n,r}(p_j)$ in the notation of that section, with $C_j \to C_{n,r}$. The zero transformation $z_\rho = i(\rho - 1/2)\log T/(2\pi)$ preserves multiplicity and sends $\rho \mapsto 1 - \bar{\rho}$ to complex conjugation. Hence Lemma 3, with $k = n$, gives for each fixed j

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq \frac{2n + 1 - C_j}{n(n+1)}.$$

Letting $j \to \infty$ proves (1); this passage requires no pair formula for a nonsmooth endpoint test. The strict variational certificate (44) and comparison with the uniform density give $n < C_{n,r} < n + r/(3n)$. Substitution proves the first bound in (2), with strict improvement for (1); $r \leq n$ gives the second. The further increment, for $n \geq 2$, is proved in Theorem 13. □

For fixed n, the constant $C_{n,r}$ is strictly increasing in r. Indeed, for $r' < r''$, evaluating $J_{n,r'}$ and $J_{n,r''}$ at the positive minimizer for r'' shows that $J_{n,r''} - J_{n,r'}$ is $(r'' - r')/n$ times a strictly positive distance integral, while $J_{n,r'} \geq C_{n,r'}$. Consequently $b_{n,r} \geq b_{n,n}$, strictly when $r < n$. Table 1 uses $r = n$ and therefore applies to every fixed field of the displayed degree. Each decimal with an ellipsis lists initial digits of the exact constant, not a rounded value or an estimate for a finite height T.

Table 1: Degree-only specializations of (1); the strict geometric increment from Theorem 13 is not included.

Degree	Degree-only distinct fraction	Percentage
2	0.44491871304792468 ...	44.4918713 ... %
3	0.30571236760745205 ...	30.5712367 ... %
4	0.23340361496350142 ...	23.3403614 ... %
5	0.18892628222294294 ...	18.8926282 ... %
6	0.15875238102308097 ...	15.8752381 ... %

Rank-sensitive examples. Every non-Galois cubic field has normal-closure group S_3 in its natural action on three embeddings: the only transitive subgroups of S_3 are C_3 and S_3, and the former gives a Galois cubic field. In the non-Galois case the point stabilizer has two orbits, of sizes one and two. Thus $r = 2$ and (1) gives the stronger fraction $b_{3,2} = 0.31488413586578248\ldots$. For a quartic field whose normal-closure group acts doubly transitively on the four embeddings, the point stabilizer is transitive on the other three embeddings. Again $r = 2$, giving $b_{4,2} = 0.24168413176770299\ldots$. At degree one, $K = \mathbb{Q}$ and $n = r = 1$; the bound is $b_{1,1} = (3 - C_{1,1})/2$, recovering Lamzouri's distinct-zero constant [1, Theorem 1.1]. The geometric increment asserted above is restricted to $n \geq 2$.

Corollary 2. *For every fixed finite abelian extension $K/\mathbb{Q}$, unconditionally,*

$$\boxed{\liminf_{T\to\infty} \frac{N_{K,0}(T)}{N_K(T)} \geq c_0 := \frac{3}{2} - \frac{1}{\sqrt{2}}\cot(1/\sqrt{2}) = 0.67250070367941164\ldots.}$$

This does not claim that these zeros are simple in ζ_K.

Proof. The primitive Dirichlet factorization is $\zeta_K(s) = \prod_{\chi\in X_K} L(s,\chi)$, with $|X_K| = n$; the trivial primitive character contributes $\zeta(s)$. All factors are holomorphic in the open critical strip. Their local zero multiplicities therefore add, including at common zeros, so

$$N_{K,0}(T) = \sum_{\chi \in X_K} N_{\chi,0}(T), \qquad N_K(T) = \sum_{\chi \in X_K} N_\chi(T),$$

where the factor counts use the same conventions and height window. For any $\varepsilon > 0$, the fixed primitive Dirichlet result [14, Theorems A–B] gives $N_{\chi,0}(T) \geq (c_0 - \varepsilon)N_\chi(T)$ for all sufficiently large T, since the multiplicity-weighted critical-line count is at least the simple-critical-line count. There are only finitely many factors, so one common threshold for T suffices. Sum these inequalities and then let $\varepsilon \downarrow 0$. Section 6 also gives the degree-one adaptation of the pair-formula argument. Neither derivation assumes that different factors have disjoint zero sets. $\square$

The trigonometric optimization constant is known in generalized Montgomery–Taylor theory: it is the reciprocal reproducing-kernel value in [28, Theorem 5], with $c_1 = n$, $c_2 = r/n$, and $\Delta = 1$. The theorem's parameter restriction holds because $r/n^2 \leq 1 < 5/3$. Corollary 2 is a factorwise consequence of the primitive Dirichlet theorem, not an independent zero-proportion theorem. The extensions developed here are the integer-threshold complex-zero certificate, its unconditional Dedekind application, and the projection-and-triples refinement that uses the height window for $n \geq 2$, with the qualifications on priority in Section 1. The GRH-dependent numerical comparisons are made in Section 8; no claim of numerical superiority over their stronger consequences is made.

3 The integer-threshold Hilbert certificate

Let $\eta \in L^2(\mathbb{R})$ be real, even, and compactly supported, with $\int_{\mathbb{R}} \eta(u)^2\, du = 1$. For compactly supported $f \in L^1(\mathbb{R})$, use the entire Fourier transform

$$\widehat{f}(z) = \int_{\mathbb{R}} f(u)e^{-2\pi i z u}\, du \quad (z \in \mathbb{C}), \qquad K_\eta(z) = \widehat{\eta^2}(z).$$

Compact support makes this extension entire. In particular, K_η is even, is real on $\mathbb{R}$, and satisfies $K_\eta(0) = 1$. Let Z be a finite nonempty multiset with support $\Omega \subset \mathbb{C}$ and positive integer multiplicities m_z. Assume that Z is invariant under complex conjugation: $\bar{z} \in \Omega$ and $m_{\bar{z}} = m_z$ for every $z \in \Omega$. Define

$$N = \sum_{z \in \Omega} m_z, \quad D = |\Omega|, \quad S = |\{x \in \Omega \cap \mathbb{R} : m_x = 1\}|, \quad N_C = \sum_{z \in \Omega \setminus \mathbb{R}} m_z,$$

and

$$E = \sum_{z,w \in \Omega} m_z m_w K_\eta(z - w)^2.$$

This is an ordered double sum with multiplicities. The square is an ordinary square, not an absolute square; individual summands need not be real. The proof below shows that the total E is real and positive. No assertion about zeta functions or an asymptotic limit is needed in this finite lemma.

Lemma 3. *For every integer $k \geq 1$,*

$$D \geq \frac{(2k+1)N - E}{k(k+1)}. \tag{3}$$

In fact,

$$D \geq \frac{(2k+1)N - E}{k(k+1)} + \frac{k-1}{k+1}S. \tag{4}$$

Moreover, the threshold defect satisfies

$$\Delta_k := E - (2k+1)N + k(k+1)D \geq N_C. \tag{5}$$

Proof. The real tensor and its energy. Use Lamzouri's functions [1, Proposition 2.1]

$$f_z(u) = \eta(u)e^{-2\pi i z u}, \quad g_z = (f_z + f_{\bar{z}})/2, \quad h_z = (f_z - f_{\bar{z}})/(2i).$$

Compact support ensures that these functions belong to $L^2(\mathbb{R};\mathbb{C})$ for every fixed $z \in \mathbb{C}$. We take its inner product to be linear in the first variable:

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}} \Phi(u)\overline{\Psi(u)}\,du, \qquad \langle f_z, f_w \rangle = K_\eta(z - \bar{w}).$$

The conjugate-linear isometric involution $J\Phi(u) = \overline{\Phi(-u)}$ has the closed real fixed space $\mathcal{H}_0 = \{\Phi : J\Phi = \Phi\}$, with real scalars. Changing u to $-u$ shows that the inner product of two elements of $\mathcal{H}_0$ is real, so this is a real Hilbert space. Since $Jf_z = f_{\bar{z}}$, the vectors f_x for real x, and all g_z, h_z, belong to $\mathcal{H}_0$. Furthermore,

$$\|f_x\|^2 = 1, \qquad \|g_z\|^2 - \|h_z\|^2 = \mathrm{Re}\langle f_z, f_{\bar{z}} \rangle = 1.$$

For $a, b \in \mathcal{H}_0$, identify the real tensor $a \otimes b$ with the operator $v \mapsto \langle v, b \rangle a$. Let $\mathcal{P}$ contain one representative from each nonreal conjugate pair. Define the real symmetric finite-rank operator

$$F = \sum_{x \in \Omega \cap \mathbb{R}} m_x f_x \otimes f_x + 2 \sum_{z \in \mathcal{P}} m_z (g_z \otimes g_z - h_z \otimes h_z).$$

In the complexified tensor product, $f_z = g_z + i h_z$ gives $F = \sum_{z \in \Omega} m_z f_z \otimes f_z$. Here tensor products are complex bilinear; this is *not* a sum of positive Hermitian rank-one operators. The tensor inner product satisfies $\langle a \otimes b, c \otimes d \rangle = \langle a, c \rangle \langle b, d \rangle$. Consequently its norm, which agrees with the Hilbert–Schmidt norm of the real operator, satisfies

$$\begin{aligned} \|F\| * \mathrm{HS}^2 &= \sum *z, w \in \Omega m_z m_w K_\eta(z - \bar{w})^2 \\ &= \sum_{z, w \in \Omega} m_z m_w K_\eta(z - w)^2 = E. \end{aligned}$$

The second equality reindexes $w \mapsto \bar{w}$ and uses equality of the two multiplicities. The real decomposition also gives

$$\operatorname{tr} F = \sum_{x \in \Omega \cap \mathbb{R}} m_x + 2 \sum_{z \in \mathcal{P}} m_z (\|g_z\|^2 - \|h_z\|^2) = N.$$

In particular $F \neq 0$ and $E > 0$, because $N > 0$.

The three-block bound and the nonreal defect. Partition real locations into $m_x \le k$ and $m_x \ge k+1$; denote their counts by r_L, r_H and masses by M_L, M_H. Let c count nonreal conjugate pairs and put $M_C = \sum_{z \in \mathcal{P}} m_z$. Thus $N = M_L + M_H + 2M_C$, $D = r_L + r_H + 2c$, and $N_C = 2M_C$. Take

$$\begin{aligned} &U = \mathrm{span}_{\mathbb{R}}(f_x : x \in \Omega \cap \mathbb{R},\ m_x \ge k+1;\ g_z : z \in \mathcal{P}), \\ &V = U + \mathrm{span}_{\mathbb{R}}(f_x : x \in \Omega \cap \mathbb{R},\ m_x \le k), \qquad W = V + \mathrm{span}_{\mathbb{R}}(h_z : z \in \mathcal{P}). \end{aligned}$$

The range of F is contained in W, and F vanishes on $W^\perp$. Choose an orthonormal basis $(\psi_j)_{j=1}^{\dim W}$ adapted to $U \subseteq V \subseteq W$, omitting dependent spanning vectors during real Gram–Schmidt. Put $\alpha_j = \langle F\psi_j, \psi_j \rangle = \langle F, \psi_j \otimes \psi_j \rangle \in \mathbb{R}$. The tensors $\psi_j \otimes \psi_j$ are orthonormal. Bessel's inequality and the trace identity give

$$E = \|F\| * \mathrm{HS}^2 \ge \sum_j \alpha_j^2, \qquad \sum_j \alpha_j = N. \tag{6}$$

Let A, B, C be the traces over U, $V \ominus U$, $W \ominus V$. If Q_B projects orthogonally onto the middle block, then

$$B = \sum *^{x \in \Omega \cap \mathbb{R}}_{m_x \le k} m_x \|Q_B f_x\|^2 - 2 \sum_{z \in \mathcal{P}} m_z \|Q_B h_z\|^2 \le M_L.$$

Indeed, Q_B annihilates every high-multiplicity real vector and every g_z, while $\|Q_B f_x\| \le \|f_x\| = 1$. For a basis vector in the last block, $\alpha_j = -2\sum_{z\in\mathcal{P}} m_z \langle h_z, \psi_j\rangle^2 \le 0$. Also $d_U := \dim U \le r_H + c$ and $d_B := \dim(V \ominus U) \le r_L$. Apply $a^2 \ge 2(k+1)a - (k+1)^2$ on the first block, $a^2 \ge 2ka - k^2$ on the second, and $a^2 \ge 2(k+1)a$ on the last, where the latter uses $a \le 0$. Summing gives

$$\begin{aligned} E &\ge 2(k+1)A - (k+1)^2 d_U + 2kB - k^2 d_B + 2(k+1)C \\ &= 2(k+1)N - (k+1)^2 d_U - k^2 d_B - 2B \\ &\ge 2(k+1)N - (k+1)^2(r_H + c) - k^2 r_L - 2M_L. \end{aligned}$$

Subtracting $(2k+1)N - k(k+1)D$ therefore gives

$$\Delta_k \ge [M_H - (k+1)r_H] + [kr_L - M_L] + 2M_C + (k^2-1)c.$$

Both brackets are nonnegative by the multiplicity ranges, and $k^2 - 1 \ge 0$. The right-hand side is therefore at least N_C, proving both (3) and (5).

The multiplicity flag and the simple-real refinement. Let r_m count real locations of multiplicity $m \ge 1$, so $S = r_1$. Starting with $U_{k+1} = U$, define, in descending order,

$$U_m = U_{m+1} + \operatorname{span}_{\mathbb{R}}(f_x : x \in \Omega \cap \mathbb{R},\ m_x = m), \quad \mathcal{B}_m = U_m \ominus U_{m+1} \quad (m = k, \dots, 1).$$

Then $U_1 = V$. Write A_m for the trace of F on $\mathcal{B}_m$ and $d_m = \dim \mathcal{B}_m \le r_m$. For $1 \le j \le k$, the orthogonal direct sum of the blocks with $m \le j$ is $U_1 \ominus U_{j+1}$. Its projection Q_j annihilates every real vector of multiplicity greater than j and every g_z. Thus

$$L_j := \sum_{m=1}^{j} A_m = \sum_{\substack{x\in\Omega\cap\mathbb{R} \\ m_x \le j}} m_x \|Q_j f_x\|^2 - 2\sum_{z\in\mathcal{P}} m_z \|Q_j h_z\|^2 \le \sum_{m=1}^{j} m r_m.$$

The individual A_m need not be nonnegative; only these cumulative upper bounds are used. The finite summation identity

$$\sum_{m=1}^{k} (k+1-m)A_m = \sum_{j=1}^{k} L_j \le \sum_{m=1}^{k} (k+1-m) m r_m$$

now controls their weighted sum. Apply $a^2 \ge 2ma - m^2$ on $\mathcal{B}_m$, retaining the previous inequalities on U and $W \ominus V$. Bessel's inequality in this finer adapted basis gives

$$\begin{aligned} E &\ge 2(k+1)N - (k+1)^2 d_U - \sum_{m=1}^{k} m^2 d_m - 2\sum_{m=1}^{k}(k+1-m)A_m \\ &\ge 2(k+1)N - (k+1)^2(r_H + c) - \sum_{m=1}^{k} (2(k+1)m - m^2) r_m. \end{aligned}$$

Subtracting $(2k+1)N - k(k+1)D$ gives the explicit remainder

$$\begin{aligned} \Delta_k \ge& \sum_{m=1}^{k} (k-m)(k+1-m) r_m + \sum_{m \ge k+1} (m-k-1) r_m \\ &+ 2M_C + (k^2-1)c \ge k(k-1)S + N_C. \end{aligned}$$

Every term in the remainder is nonnegative; the coefficient of $r_1 = S$ is $k(k-1)$. Dividing its S contribution by $k(k+1)$ proves (4). All arguments remain valid when blocks are empty or spanning vectors are dependent: dimensions can only decrease, and empty traces and sums are zero. $\square$

Choice of threshold and comparison with rank–trace. The case $k = 1$ in (3) recovers Lamzouri's distinct-element inequality. Since $\dim W \le D \le N$, (6) and Cauchy–Schwarz give

$$N^2 \le (\dim W) \sum_j \alpha_j^2 \le DE.$$

Thus $D \ge N^2/E$ and $E/N \ge N/D \ge 1$. If $E/N \le C$ for a real constant C, necessarily $C \ge 1$. The basic bound (3), with E replaced by CN, becomes $D/N \ge B_k(C)$, where

$$B_k(C) = \frac{2k+1-C}{k(k+1)}, \qquad B_{k+1}(C) - B_k(C) = \frac{2(C-k-1)}{k(k+1)(k+2)}.$$

The sign of this difference shows that $k = \lfloor C \rfloor$ is a maximizer over integers $k \ge 1$. It is unique for nonintegral C; at an integer $C = m \ge 2$ the two maximizers are $m-1$ and m. For $C = 1$ the unique admissible maximizer is $k = 1$. On $k \le C \le k+1$, comparison with the rank–trace consequence $D/N \ge 1/C$ gives

$$B_k(C) - \frac{1}{C} = \frac{(C-k)(k+1-C)}{Ck(k+1)} \ge 0.$$

The improvement is strict for nonintegral C and vanishes at integers when an optimal threshold is used. This optimization concerns the basic bound (3); optimizing a bound that also retains S or N_C can depend on those additional data.

Why integer multiplicities matter. The lemma does not extend to arbitrary positive real weights. For a single real location with weight $m = k + 1/2$, normalization gives $N = m$, $D = 1$, and $E = m^2$, so

$$E - (2k+1)N + k(k+1)D = (m-k)(m-k-1) = -\tfrac{1}{4}.$$

This would contradict (3). Positive integer multiplicities, as supplied by zero orders, are essential.

4 An unconditional pair formula for every fixed field

We give six quantitative propositions and the analytic preliminaries on which they depend. Throughout this section the number field K is fixed, $n = [K : \mathbb{Q}]$, $d_K = |\operatorname{disc} K|$, and (r_1, r_2) is its signature, so $n = r_1 + 2r_2$. Constants denoted by O_K are independent of all the displayed real variables in their stated ranges. They need not be uniform in the field. Unless a height restriction is displayed, a sum over $\rho = \beta + i\gamma$ runs over all nontrivial zeros of ζ_K, including real zeros. Every zero sum counts analytic multiplicity, and every pair sum is over ordered pairs, including the diagonal. Thus $N_K(T)$ counts $0 < \gamma \le T$, with multiplicity, not both signs of the ordinate. For compactly supported integrable f, our Fourier convention and its entire extension are

$$\widehat{f}(z) = \int_{\mathbb{R}} f(u) e^{-2\pi i u z}\, du, \qquad z \in \mathbb{C}, \qquad W(z) = \frac{4}{4 - z^2}.$$

Compact support justifies differentiation under the Fourier integral at every complex argument. The function W is meromorphic, but its poles at ± 2 are never reached by a zero difference, since $|\operatorname{Re}(\rho - \rho')| < 1$.

Lemma 4 (Quantitative strict-support pair formula)**.** *Let $0 < b < 1$ and let $f \in C_c^2(\mathbb{R})$ be real and even, with* $\operatorname{supp} f \subset [-b, b]$. *Put $L = \log T$ and*

$$\mathcal{R}_b(T) = L^{-1} + T^{b-1} L^2 + T^{(b-1)/2} L^{-1/2} + T^{-1/2}.$$

As $T \to \infty$, without GRH or Artin holomorphy,

$$\sum_{0<\gamma,\gamma'\le T} \widehat{f}\left(i(\rho-\rho')\frac{L}{2\pi}\right) W(\rho-\rho')$$
$$= N_K(T)\left(nf(0) + \frac{r}{n}\int_{-1}^{1} |u| f(u)\,du + O_{K,b,f}(\mathcal{R}_b(T))\right), \tag{7}$$

where $r = |H\backslash G/H|$ as in Section 2. For each fixed $b < 1$, the displayed error is also $O_{K,b,f}(1/\log T)$. The zeros in the sum are their actual complex locations. The formula holds for arbitrary real T sufficiently large, including heights that are zero ordinates.

The proof of this lemma is completed in Proposition 11. The restriction $b < 1$ is used explicitly in that integration of the sharp-window error. Corollary 12 subsequently extends the formula to globally C^2 endpoint tests by a different, boundary-vanishing estimate. No nonsmooth endpoint assertion is needed.

4.1 Completed zeta function and an unconditional local zero count

Write

$$Z_K(s) = \frac{\zeta_K'}{\zeta_K}(s), \qquad -Z_K(s) = \sum_{m\ge 2} A_K(m) m^{-s} \quad (\operatorname{Re} s > 1).$$

The Euler product gives $0 \le A_K(m) \le n\Lambda(m)$; its full local calculation is given in Proposition 6 below. Set

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s),$$
$$\xi_K(s) = s(s-1) d_K^{s/2} \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s).$$

The classical unconditional facts used here are that ξ_K is entire of order one, is nonzero at 0 and 1, has precisely the nontrivial zeros of ζ_K, all in $0 < \operatorname{Re} s < 1$, and satisfies

$$\xi_K(s) = \xi_K(1-s), \qquad \xi_K(\bar{s}) = \overline{\xi_K(s)}.$$

See [13, Section 2.1, equations (2.1)–(2.2)] and [18, Section 2.1] for these completed-function facts and the trivial-zero multiplicities. The gamma duplication formula identifies the gamma factors in [13] with the ones above; a constant normalization of ξ_K does not affect its logarithmic derivative. Only these unconditional completed-function facts are imported: in particular, we do not use the GRH-dependent pair formula in [18]. The local zero estimate below is proved before the global normalization, so the two estimates do not depend circularly on one another.

Lemma 5 (Local zero count)**.** *For every real t,*

$$U_K(t) := \sum_{\rho} \frac{2-\beta}{(2-\beta)^2 + (t-\gamma)^2} = \frac{n}{2}\log(|t|+2) + O_K(1), \tag{8}$$
$$\#\{\rho : |\gamma - t| \le 1\} \ll_K \log(|t|+2), \tag{9}$$
$$\sum_{\rho} \frac{1}{1+(t-\gamma)^2} \ll_K \log(|t|+2). \tag{10}$$

The series in these statements converge absolutely.

Proof. Order one and Jensen's formula give $\#\{\rho : |\rho| \le R\} = O_K(R^{5/4})$, and consequently $\sum_\rho |\rho|^{-2} < \infty$. Hadamard factorization therefore gives

$$\frac{\xi_K'}{\xi_K}(s) = b_K + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right),$$

with normally convergent combined terms off the zeros. Since $0 < \beta < 1$, the real parts of the two sums separately converge absolutely at any fixed s off the zeros. Hence

$$\mathrm{Re}\,\frac{\xi_K'}{\xi_K}(\sigma+it) = c_K + \sum_\rho \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}, \qquad c_K = \mathrm{Re}\, b_K + \sum_\rho \mathrm{Re}(1/\rho).$$

In fact $c_K = 0$. The functional equation makes the logarithmic derivatives at 2 and -1 negatives. Reindexing the sum at -1 by $\rho \mapsto 1-\bar\rho$ makes it the negative of the sum at 2. Adding the two displayed identities gives $2c_K = 0$.

At $s = 2+it$, logarithmic differentiation of the completion now gives

$$\begin{aligned} U_K(t) = \mathrm{Re}\Bigg(&\frac{1}{s} + \frac{1}{s-1} + \frac{1}{2}\log d_K - \frac{r_1}{2}\log\pi - r_2\log(2\pi) \\ &+ \frac{r_1}{2}\psi(s/2) + r_2\psi(s) + Z_K(s) \Bigg), \qquad \psi = \Gamma'/\Gamma. \end{aligned}$$

Here $|Z_K(2+it)| \le n\sum_m \Lambda(m)m^{-2}$ uniformly in t. Stirling's estimate for ψ on these fixed vertical lines, together with continuity for bounded t, proves (8). All summands are positive. If $|\gamma - t| \le 1$, each is at least $1/5$, because $1 < 2-\beta < 2$. This proves (9). Finally, for $a \in (1,2)$,

$$\frac{a}{a^2+y^2} \ge \frac{1}{4+y^2} \ge \frac{1}{4(1+y^2)}.$$

Summation and (8) prove (10). □

4.2 The arithmetic second moment

Proposition 6 (Permutation-rank coefficient)**.** *Uniformly for $y \ge 2$,*

$$B_K(y) := \sum_{m \le y} A_K(m)^2 = ry\log y + O_K(y). \tag{11}$$

For every rational prime p, including ramified primes, and every $j \ge 1$,

$$A_K(p^j) = \log p \sum_{\substack{\mathfrak{p}|p \\ f_\mathfrak{p}|j}} f_\mathfrak{p}, \qquad 0 \le A_K(m) \le n\Lambda(m). \tag{12}$$

Here $f_\mathfrak{p}$ is the residue degree. All coefficients outside prime powers vanish.

Proof. The local Euler factor at p is $\prod_{\mathfrak{p}|p}(1-p^{-f_\mathfrak{p}s})^{-1}$. Its negative logarithmic derivative is

$$\sum_{\mathfrak{p}|p} f_\mathfrak{p} \log p \sum_{k\ge1} p^{-kf_\mathfrak{p}s}.$$

This proves the exact coefficient formula. Since $\sum_{\mathfrak{p}|p} e_\mathfrak{p} f_\mathfrak{p} = n$ and $e_\mathfrak{p} \ge 1$, it also proves the bound in (12) without a ramification restriction.

Let M be the normal closure, $G = \mathrm{Gal}(M/\mathbb{Q})$, $H = \mathrm{Gal}(M/K)$, and $\Omega = G/H$. Write $P(g) = \#\,\mathrm{Fix}_\Omega(g)$. At a prime unramified in M, the cycle lengths of Frobenius on Ω are the residue degrees of the primes of K above p. Consequently

$$A_K(p) = P(\mathrm{Frob}*p)\log p.$$

Burnside's lemma, applied to the diagonal action on Ω^2, gives

$$\frac{1}{|G|}\sum *g \in GP(g)^2 = \#(G\backslash\Omega^2) = \#(H\backslash G/H) = r. \tag{13}$$

The second equality follows by moving the first coordinate to H; its stabilizer is H, acting on the second coordinate by left multiplication.

For a conjugacy class $C \subset G$, let $\pi_C(y)$ count the rational primes at most y, unramified in M, with Frobenius in C. The fixed-extension consequence of effective Chebotarev [29, Theorem 1.1] (or [30, Theorem 1.1]) is

$$\pi_C(y) = \frac{|C|}{|G|}\,\mathrm{Li}(y) + O_K\left(ye^{-c_K\sqrt{\log y}}\right) \tag{14}$$

for some $c_K > 0$. The finitely many classes permit one choice of constants. Any exceptional term has the form $O_K(\mathrm{Li}(y^{\beta_0}))$ with fixed $\beta_0 < 1$; it is absorbed in (14), since $y^{\beta_0 - 1}e^{c_K\sqrt{\log y}}$ is bounded for fixed K. Any lower threshold on y in the cited effective theorem is absorbed by enlarging the fixed-field constant. If $M = \mathbb{Q}$, use the ordinary prime number theorem in the same form.

Partial summation with the weight $\log^2 v$ gives

$$\begin{aligned}\sum_{\substack{p\le y\\ \mathrm{Frob}_p\in C}} \log^2 p &= \pi_C(y)\log^2 y - 2\int_2^y \pi_C(v)\frac{\log v}{v}\,dv\\ &= \frac{|C|}{|G|}\int_2^y \log v\,dv + O_K(y)\\ &= \frac{|C|}{|G|}y\log y + O_K(y).\end{aligned}$$

The lower-endpoint constant is harmless. For the error, the boundary term is $O_K(y\log^2 y\, e^{-c_K\sqrt{\log y}}) = O_K(y)$, and the integral is bounded by a constant times $\int_2^y \log v\, e^{-c_K\sqrt{\log v}}\,dv = O_K(y)$. Multiply by the class value $P(C)^2$, sum over C, and use (13). This evaluates the first powers of unramified primes with leading coefficient r.

For each of the finitely many primes ramified in M,

$$\sum_{p^j\le y} A_K(p^j)^2 \le n^2(\log p)^2\left\lfloor\frac{\log y}{\log p}\right\rfloor = O_K(\log y).$$

The total contribution of higher prime powers is bounded by

$$n^2\sum_{2\le j\le\log_2 y}\ \sum_{p\le y^{1/j}}(\log p)^2 \ll_K y^{1/2}\log^3(2y) = O_K(y).$$

These estimates prove (11). $\square$

A tensor-product interpretation. There is also an arithmetic check using the finite etale algebra

$$K \otimes_{\mathbb{Q}} K \simeq \prod_{HgH \in H\backslash G/H} K\, g(K) = \prod_{HgH} M^{H \cap gHg^{-1}}.$$

Indeed, if $K = \mathbb{Q}(\theta)$, factor the separable minimal polynomial of θ over K and apply the Chinese remainder theorem. Its irreducible factors correspond to the H-orbits on G/H; the associated map is $a \otimes b \mapsto a\, g(b)$. Its embeddings into M are identified with the ordered pairs of embeddings of K, so the embedding character of this algebra is P^2. If A_E denotes the sum of the logarithmic-derivative coefficients of its field factors, then at every prime unramified in M,

$$A_E(p) \log p = A_K(p)^2.$$

For completeness, the fixed-field prime ideal theorem gives $\sum_{m \le y} A_E(m) = ry + O_K(y e^{-c\sqrt{\log y}})$ for some $c > 0$, with any fixed exceptional zero absorbed as above. The terms with $m = p^j$, $j \ge 2$, contribute $O_K(y^{1/2} \log^2(2y))$. Removing them and the finitely many ramified primes, and then applying partial summation with weight $\log p$, gives the same leading term $ry \log y + O_K(y)$ for $\sum_{p \le y} A_E(p) \log p$. The coefficient identity is asserted only at unramified primes; the preceding error estimate handles the omitted primes. For example, for a non-Galois cubic field, $G \simeq S_3$ and H is a point stabilizer of order two. Its two orbits on G/H have sizes one and two; the corresponding intersections are H and the trivial group. Hence $K \otimes_{\mathbb{Q}} K \simeq K \times M$ and $r = 2$.

4.3 An exact resolvent formula

Define the fixed constant

$$q_K = \frac{1}{2} \log d_K - \frac{r_1}{2} \log \pi - r_2 \log(2\pi), \qquad \psi = \Gamma'/\Gamma.$$

Proposition 7 (Resolvent and uniform residual)**.** *For $x \ge 1$ and $t \in \mathbb{R}$, put*

$$H_x(t) = 2 \sum_{\rho} \frac{x^{\rho - 1/2 - it}}{1 - (\rho - 1/2 - it)^2}, \qquad P_x(t) = \sum_{m \ge 2} \frac{A_K(m)}{m^{1/2+it}} \min(m/x, x/m), \tag{15}$$

$$\begin{aligned} G_K(t) &= 2q_K + \frac{r_1}{2}\{\psi(-1/4 + it/2) + \psi(3/4 - it/2)\} \\ &\quad + r_2\{\psi(-1/2 + it) + \psi(3/2 - it)\} + Z_K(3/2 - it), \qquad (16) \\ B_x(t) &= x^{-1} G_K(t). \end{aligned}$$

All nontrivial zeros, including real zeros, occur in H_x. There is an exact identity

$$H_x(t) = -P_x(t) + B_x(t) + R_x(t), \tag{17}$$

where, with $b_0 = r_1 + r_2 - 1$ and $b_j = r_2 + r_1 \mathbf{1}_{2|j}$ for $j \ge 1$,

$$R_x(t) = \frac{2x^{1/2-it}}{(1/2+it)(3/2-it)} + 2 \sum_{j \ge 0} \frac{b_j x^{-j-1/2-it}}{(j - 1/2 + it)(j + 3/2 + it)}. \tag{18}$$

The zero, prime, and trivial-zero series converge absolutely and locally uniformly on $[1, \infty) \times \mathbb{R}$. Uniformly on this domain,

$$G_K(t) = n \log(|t| + 2) + O_K(1), \tag{19}$$

$$R_x(t) \ll_K \frac{x^{1/2}}{1+t^2} + \frac{x^{-1/2}}{|t|+2}, \qquad \int_{\mathbb{R}} |R_x(t)|^2 \, dt \ll_K x. \tag{20}$$

Proof. We use a rational Mellin kernel, whose quadratic decay permits direct contour integration. Here $\int_{(d)}$ denotes the upward-oriented integral on $\operatorname{Re} w = d$, and powers of positive real numbers use their real logarithms. For $d > 1$ and $y > 0$,

$$\frac{1}{2\pi i}\int_{(d)} \frac{2y^w}{1-w^2}\,dw = \begin{cases} y^{-1} - y, & y > 1, \\ 0, & 0 < y \le 1. \end{cases} \tag{21}$$

To verify this, shift the line left for $y \ge 1$: the residues at $w = 1, -1$ are $-y, y^{-1}$. On $\operatorname{Re} w = -R$, $R > 2$, the absolute integral is $O(y^{-R}/R)$, including when $y = 1$. For $y < 1$, shift right; there are no residues and the new integral is $O(y^R/R)$. For each fixed shifted line, the horizontal integrals vanish because the rational factor decays quadratically. This proves (21) with its value at $y = 1$.

Consider the absolutely convergent integral

$$I_x(t) = \frac{1}{2\pi i}\int_{(2)} -Z_K(z)\frac{2x^{z-1/2-it}}{1-(z-1/2-it)^2}\,dz. \tag{22}$$

The Dirichlet series on this line is absolutely convergent, so Fubini and (21), with $w = z - 1/2 - it$, give

$$I_x(t) = \sum_{m<x} A_K(m) m^{-1/2-it}(m/x - x/m). \tag{23}$$

The term $m = x$ is exactly zero; thus no half-weight convention enters.

Justification of the contour shift. First fix an integer $M \ge 1$ and shift to $\operatorname{Re} z = -M - 1/2$. Here are bounds for both types of added contour. Order one, already used above, gives $N_{\mathrm{rad}}(R) := \#\{\rho : |\rho| \le R\} = O_K(R^{5/4})$. In each large interval $[Y, 2Y]$, remove intervals of radius $c_K Y^{-1/4}$ around γ and $-\gamma$ for all zeros with $|\rho| \le 5Y$. Their total length is less than Y for sufficiently small fixed $c_K > 0$. Choose a remaining height U. Both U and $-U$ are then at distance at least $c_K Y^{-1/4}$ from every zero ordinate: zeros outside the removed set have $|\gamma| \ge \sqrt{25Y^2 - 1} > 4Y$, whereas $U \le 2Y$. For Y sufficiently large in terms of M, all points $z = \sigma \pm iU$, $-M - 1/2 \le \sigma \le 2$, satisfy $|z| \le 3Y$. Partial summation of $N_{\mathrm{rad}}(R) = O_K(R^{5/4})$ gives $\sum_{|\rho|\le 5Y} |\rho|^{-1} = O_K(1 + Y^{1/4})$; the absence of a zero at 0 handles its lower endpoint. In the Hadamard logarithmic derivative the zeros with $|\rho| \le 5Y$ therefore contribute at most

$$O_K\left(Y^{1/4} N_{\mathrm{rad}}(5Y) + \sum_{|\rho|\le 5Y} |\rho|^{-1}\right) = O_K(Y^{3/2}).$$

For $|\rho| > 5Y$ and sufficiently large Y (depending on M),

$$\left|\frac{1}{z-\rho} + \frac{1}{\rho}\right| \ll \frac{Y}{|\rho|^2}, \qquad \sum_{|\rho|>5Y} |\rho|^{-2} = O_K(Y^{-3/4}).$$

Thus $\xi_K'/\xi_K(z) = O_{K,M}(Y^{3/2})$ on these horizontal segments. Subtracting the rational and gamma logarithmic derivatives, which are $O_{K,M}(\log Y)$ there, gives the same bound for $Z_K(z)$. The kernel in (22) is $O_t(Y^{-2})$ on those segments; their integrals are therefore $O_{K,M,x,t}(Y^{-1/2})$ and tend to zero.

For the left vertical line we need a bound uniform in M. Logarithmic differentiation of the completed functional equation gives

$$\begin{aligned} -Z_K(s) = Z_K(1-s) + 2q_K + \frac{r_1}{2}\{\psi(s/2) + \psi((1-s)/2)\} \\ + r_2\{\psi(s) + \psi(1-s)\}. \end{aligned} \tag{24}$$

There are no rational terms in this identity: the contributions of $s(s-1)$ cancel because

$$\left(\frac{1}{s}+\frac{1}{s-1}\right)+\left(\frac{1}{1-s}+\frac{1}{-s}\right)=0.$$

At $s=-M-1/2+iu$, the first term on the right is bounded by its absolutely convergent Dirichlet series. The reflection identity $\psi(z)=\psi(1-z)-\pi\cot(\pi z)$ moves every negative-real-part gamma argument to a right half-plane. The cotangents are bounded uniformly: the real parts of s and $s/2$ are half-integers and quarter-integers bounded away from integers. Stirling's estimate then gives

$$|Z_K(-M-1/2+iu)| \ll_K \log(M+|u|+2).$$

On that line, putting $v=u-t$,

$$|1-(z-1/2-it)^2|=\sqrt{(M+2)^2+v^2}\sqrt{M^2+v^2}\geq M^2+v^2.$$

Consequently the absolute left integral is at most

$$C_K x^{-M-1}\int_{\mathbb{R}}\frac{\log(M+|u|+2)}{M^2+(u-t)^2}\,du \ll_K x^{-M-1}\frac{\log(M+|t|+2)}{M},$$

by substituting $u=t+Mv$. It tends to zero as $M\to\infty$, even when $x=1$. The limits are taken first in the horizontal height at fixed M,x,t, and then in M.

Residues and prime coefficients. The poles crossed, with their residues in (22), are

z	residue
ρ	$-2m_\rho x^{\rho-1/2-it}/(1-(\rho-1/2-it)^2)$
1	$2x^{1/2-it}/((1/2+it)(3/2-it))$
$-j$	$2b_j x^{-j-1/2-it}/((j-1/2+it)(j+3/2+it))$
$3/2+it$	$xZ_K(3/2+it)$
$-1/2+it$	$-x^{-1}Z_K(-1/2+it)$.

In the first row each distinct zero is represented once with its multiplicity m_ρ; equivalently the sum defining H_x repeats it m_ρ times. The last two poles come from the rational kernel and do not coincide with zeros or poles of ζ_K. A row at $-j$ is absent when $b_j=0$. For $j\geq 1$, the functional equation and the Euler product give $\xi_K(-j)=\xi_K(1+j)\neq 0$. Thus the trivial order of vanishing at $-j$ equals the total pole order of the gamma factors, namely $b_j=r_2+r_1\mathbf{1}_{2|j}$. At 0, $\xi_K(0)\neq 0$, the gamma factors have total pole order r_1+r_2, and $s(s-1)$ has a simple zero; this gives $b_0=r_1+r_2-1$.

Both zero-residue sums converge absolutely: the nontrivial tail has a constant multiple of $\sum_\rho(1+|\rho|^2)^{-1}$ as a majorant, and the trivial tail is $O_K(\sum_{j\geq 1}j^{-2})$. The residue theorem and the preceding contour bounds give

$$I_x(t)=-H_x(t)+R_x(t)+xZ_K(3/2+it)-x^{-1}Z_K(-1/2+it).$$

For $m<x$, the coefficient in $-I_x+xZ_K(3/2+it)$ is

$$A_K(m)m^{-1/2-it}(x/m-m/x-x/m)=-A_K(m)m^{-1/2-it}(m/x).$$

For $m>x$ it is $-A_K(m)m^{-1/2-it}(x/m)$; at $m=x$ only the logarithmic derivative contributes, with coefficient $-A_K(m)m^{-1/2-it}$. Therefore

$$-I_x(t)+xZ_K(3/2+it)=-P_x(t).$$

Equation (24) at $s=-1/2+it$ identifies $-Z_K(-1/2+it)=G_K(t)$ exactly. This proves (17).

Uniform estimates and convergence. The term $Z_K(3/2-it)$ in (16) is bounded by $\sum_m A_K(m)m^{-3/2}$. For large $|t|$, Stirling gives

$$\psi(-1/4+it/2)+\psi(3/4-it/2)=2\log(|t|/2)+O(|t|^{-1}),$$
$$\psi(-1/2+it)+\psi(3/2-it)=2\log|t|+O(|t|^{-1}).$$

The imaginary parts of the logarithms cancel to this accuracy. For bounded t, all arguments stay away from gamma poles. These facts prove (19), with a possibly complex bounded error. Furthermore,

$$|(1/2+it)(3/2-it)|\geq \tfrac{1}{2}(1+t^2),$$
$$|(j-1/2+it)(j+3/2+it)|\geq \tfrac{1}{4}((j+1)^2+t^2) \qquad (j\geq 0).$$

Since $b_j\leq n$ and $x^{-j}\leq 1$,

$$|R_x(t)|\ll_K \frac{\sqrt{x}}{1+t^2}+x^{-1/2}\sum_{k\geq 1}\frac{1}{k^2+t^2}\ll_K \frac{\sqrt{x}}{1+t^2}+\frac{x^{-1/2}}{|t|+2}.$$

Squaring and integrating gives $\int_{\mathbb{R}}|R_x|^2\ll_K x+x^{-1}\ll_K x$. On compact x,t ranges, the zero, prime, and trivial-zero tails have summable majorants proportional to $(1+\gamma^2)^{-1}$, $A_K(m)m^{-3/2}$, and $(j+1)^{-2}$, respectively. This proves local uniform convergence, including at $x=1$. The contour proof itself already covers that endpoint. □

There is an independent endpoint check. Taking the difference of the Hadamard logarithmic derivatives at $3/2+it$ and $-1/2+it$ gives

$$H_1(t)=\frac{\xi_K'}{\xi_K}(3/2+it)-\frac{\xi_K'}{\xi_K}(-1/2+it)=2\operatorname{Re}\frac{\xi_K'}{\xi_K}(3/2+it).$$

The combined reciprocal series is absolutely convergent, and the last equality follows from the functional equation and conjugation. No differentiation in x at the endpoint is used.

The bounded term $Z_K(3/2-it)$ in (16) is essential for the stated pointwise residual. For example, absolute convergence gives

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|Z_K(3/2-it)|^2\,dt=\sum_{m\geq 2}\frac{A_K(m)^2}{m^3}>0.$$

This follows by termwise integration of an absolutely summable double series. In particular that term cannot be placed in an $O_K((|t|+2)^{-1})$ remainder. It is included in G_K throughout.

4.4 Plancherel and the sharp height window

Proposition 8 (Exact kernel and localization)**.** *For $T\geq 3$ and $x>0$, define*

$$A_T=\{\rho:0<\gamma\leq T\},\qquad F_K(x,T)=\sum_{\rho,\rho'\in A_T}x^{\rho-\rho'}W(\rho-\rho').$$

Here A_T is a multiset, so $|A_T|=N_K(T)$. For every $x>0$, define the finite sum

$$H_{x,T}(t)=2\sum_{\rho\in A_T}\frac{x^{\rho-1/2-it}}{1-(\rho-1/2-it)^2}.$$

Then

$$F_K(x,T) = \frac{1}{2\pi}\int_{\mathbb{R}} |H_{x,T}(t)|^2\,dt \ge 0. \tag{25}$$

Uniformly for $x \ge 1$ and $T \ge 3$,

$$F_K(x,T) = \frac{1}{2\pi}\int_0^T |H_x(t)|^2\,dt + O_K(x\log^3(T+2)). \tag{26}$$

The same local estimates also give the normalization

$$N_K(T) = \frac{n}{2\pi}T\log T + O_K(T). \tag{27}$$

No condition that T avoid a zero ordinate is imposed.

Proof. The finite-window identity. For $\delta \in (-1/2, 1/2)$ and $\gamma \in \mathbb{R}$ put $k_{\delta,\gamma}(t) = [1 + (t-\gamma+i\delta)^2]^{-1}$. Its Fourier transform is exactly

$$\widehat{k}_{\delta,\gamma}(v) = \pi e^{-2\pi|v|}e^{-2\pi\delta v}e^{-2\pi i\gamma v}. \tag{28}$$

Indeed, in the Fourier integral set $w = t-\gamma+i\delta$. The exponential becomes $e^{-2\pi iwv}e^{-2\pi i\gamma v}e^{-2\pi\delta v}$. Shift the line $\operatorname{Im} w = \delta$ to the real axis; it crosses no pole at $w = \pm i$, and the short vertical integrals tend to zero by quadratic denominator decay. The remaining transform of $(1+w^2)^{-1}$ is $\pi e^{-2\pi|v|}$, by residues. This also verifies the sign of the factor involving δ.

For $\rho = \beta + i\gamma$ set $\delta_\rho = \beta - 1/2$. Remove the common factor of absolute value one before applying (28):

$$e^{it\log x}H_{x,T}(t) = 2\sum_{\rho\in A_T} x^{\rho-1/2}k_{\delta_\rho,\gamma}(t).$$

This is a finite sum of $L^1 \cap L^2$ functions. Plancherel gives, with $a_{\rho,\rho'} = \rho + \bar{\rho}' - 1$,

$$\int_{\mathbb{R}} |H_{x,T}(t)|^2\,dt = 4\pi^2 \sum_{\rho,\rho'\in A_T} x^{a_{\rho,\rho'}}\int_{\mathbb{R}} e^{-4\pi|v|}e^{-2\pi a_{\rho,\rho'}v}\,dv.$$

Because $|\operatorname{Re} a_{\rho,\rho'}| < 1$, each integral converges absolutely, and

$$\int_{\mathbb{R}} e^{-4\pi|v|}e^{-2\pi av}\,dv = \frac{1}{2\pi(2+a)} + \frac{1}{2\pi(2-a)} = \frac{W(a)}{2\pi}.$$

The involution $J\rho = 1-\bar{\rho}$ preserves both ordinate and multiplicity. Replacing ρ' by $J\rho'$ converts $a_{\rho,\rho'}$ to $\rho - \rho'$, without changing A_T. This proves (25), including its constant $1/(2\pi)$. Positivity is a property of the whole pair sum, not of its individual complex summands. The same argument underlies [11, Lemma 3]; the calculation above includes the complex shifts and the exact positive-height window.

Bounds for both infinite tails. Let B_T be the multiset of excluded nontrivial zeros, including all real zeros, and define

$$S_A(t) = \sum_{\rho\in A_T}\frac{1}{1+(t-\gamma)^2}, \qquad S_B(t) = \sum_{\rho\in B_T}\frac{1}{1+(t-\gamma)^2}, \qquad \ell = \log(T+2).$$

We shall use

$$\begin{aligned} &S_A(t) \ll_K \ell \quad (0 \le t \le T),\\ &S_B(t) \ll_K \ell\left(\frac{1}{1+t} + \frac{1}{1+T-t}\right) \quad (0 \le t \le T),\\ &S_A(t) \ll_K \frac{\ell}{1+\operatorname{dist}(t,[0,T])} \quad (t \notin [0,T]). \end{aligned} \tag{29}$$

For $a \ge 0$ and $J \ge 2$, integral comparison gives

$$\sum_{j\ge 0} \frac{1}{1+(a+j)^2} \ll \frac{1}{1+a}, \qquad \sum_{j>J} \frac{\log(j+3)}{j^2} \ll \frac{\log(J+3)}{J}. \tag{30}$$

The first line of (29) follows from (10). For the second, partition excluded ordinates into $(-j-1,-j]$ and $(T+j, T+j+1]$, $j \ge 0$. Lemma 5 bounds their contributions by

$$C_K \sum_{j\ge 0} \frac{\log(j+3)}{1+(t+j)^2}, \qquad C_K \sum_{j\ge 0} \frac{\log(T+j+3)}{1+(T-t+j)^2},$$

respectively. Take $J = \lceil T+3 \rceil$. Up to J the logarithms are $O(\ell)$, so the first estimate in (30) applies. Beyond J the denominator is at least j^2 and both logarithms are $O(\log(j+3))$. The second estimate bounds this tail by $O_K(\ell/(T+1))$, which is bounded by either required endpoint majorant. This justifies the treatment of zeros at arbitrarily large heights.

For $t < 0$, partition the inside ordinates into $(j, j+1] \cap (0,T]$; for $t > T$, use $(T-j-1, T-j] \cap (0,T]$. Each interval has $O_K(\ell)$ zeros. Writing $d = \operatorname{dist}(t,[0,T])$, its kernel denominator is at least $1+(d+j)^2$. Summing and using (30) proves the third line of (29).

For real y and $|\delta| < 1/2$,

$$|1+(y+i\delta)^2| \ge 1+y^2-\delta^2 \ge \tfrac{3}{4}(1+y^2).$$

Consequently, for $x \ge 1$,

$$|H_{x,T}(t)| \le \tfrac{8}{3}\sqrt{x}\, S_A(t), \qquad |H_x(t) - H_{x,T}(t)| \le \tfrac{8}{3}\sqrt{x}\, S_B(t).$$

The three integrated estimates needed to compare squared norms are

$$\int_0^T S_A S_B \ll_K \ell^3, \qquad \int_0^T S_B^2 \ll_K \ell^2, \qquad \int_{\mathbb{R}\setminus[0,T]} S_A^2 \ll_K \ell^2. \tag{31}$$

For the first, integrate the endpoint factors $(1+t)^{-1}$ and $(1+T-t)^{-1}$ in (29). For the other two, their squares have bounded integrals. If $H_{x,B} = H_x - H_{x,T}$, then the exact difference of norms is

$$\begin{aligned}\int_{\mathbb{R}} |H_{x,T}|^2 - \int_0^T |H_x|^2 = \int_{\mathbb{R}\setminus[0,T]} |H_{x,T}|^2 \\ - \int_0^T \{2\operatorname{Re}(H_{x,T}\overline{H_{x,B}}) + |H_{x,B}|^2\}.\end{aligned}$$

Taking absolute values and applying (31) gives $O_K(x\ell^3)$, which proves (26). The intervals used above include a zero at T in A_T and a real zero in B_T; no boundary zero is omitted.

Normalization by integrating the same kernel. For a zero ρ put $a_\rho = 2-\beta \in (1,2)$ and

$$h_\rho(t) = \frac{a_\rho}{a_\rho^2 + (t-\gamma)^2}, \qquad \int_{\mathbb{R}} h_\rho(t)\, dt = \pi.$$

If $0 < \gamma \le T$, direct integration of each tail gives

$$0 \le \pi - \int_0^T h_\rho(t)\, dt \ll \frac{1}{1+\gamma} + \frac{1}{1+T-\gamma}.$$

Unit-interval summation using Lemma 5 shows that the sum of these losses over A_T is bounded by

$$C_K \ell \sum_{0\le j\le T+1} (1+j)^{-1} = O_K(\ell^2).$$

For excluded zeros, the bound $h_\rho(t) \le 2[1+(t-\gamma)^2]^{-1}$ gives

$$\sum_{\rho \in B_T} \int_0^T h_\rho(t)\,dt \le 2\int_0^T S_B(t)\,dt \ll_K \ell^2.$$

All terms here are nonnegative, so Tonelli's theorem justifies each interchange. We obtain the useful identity

$$\int_0^T U_K(t)\,dt = \pi N_K(T) + O_K(\ell^2). \tag{32}$$

Integrating (8) gives $\int_0^T U_K = (n/2)T\log T + O_K(T)$. This proves (27) directly. In particular the normalization has been derived for the positive-height count used here, with the real zeros accounted for in the excluded mass. □

4.5 The three mean-square estimates

For the remainder of this section, $\langle U, V\rangle_2 = \int_0^T U(t)\overline{V(t)}\,dt$ and $\|U\|_2^2 = \langle U, U\rangle_2$.

Proposition 9 (Mean-square inputs)**.** *Uniformly for $x \ge 1$ and $T \ge 3$, writing $L = \log T$,*

$$\|P_x\|_2^2 = Tr\log x + O_K(T + x\log(2x)), \tag{33}$$

$$\|B_x\| * 2^2 = n^2Tx^{-2}L^2 + O_K(Tx^{-2}L), \tag{34}$$

$$\|R_x\|2^2 \ll_K x. \tag{35}$$

In particular, for $1 \le x \le T$ and $u = \log(2x)$,

$$\begin{aligned} |\langle P_x, B_x\rangle_2| &\ll_K TL\sqrt{u}/x, \\ |\langle P_x, R_x\rangle_2| &\ll_K \sqrt{Txu}, \\ |\langle B_x, R_x\rangle_2| &\ll_K \sqrt{T/x}\,L. \end{aligned} \tag{36}$$

Proof. Write

$$a_x(m) = A_K(m)m^{-1/2}\min(m/x, x/m), \quad D(x) = \sum_m a_x(m)^2, \quad V(x) = \sum_m m a_x(m)^2.$$

We first evaluate the two coefficient sums. With $B_K(1) = 0$, extend (11) to $v \ge 1$ by enlarging its constant on $[1,2]$. Stieltjes partial summation gives

$$\begin{aligned} D(x) = x^{-2}&\left(xB_K(x) - \int_1^x B_K(v)\,dv\right) \\ &+ x^2\left(-x^{-3}B_K(x) + 3\int_x^\infty B_K(v)v^{-4}\,dv\right). \end{aligned}$$

The first sum uses $m \le x$ and the second $m > x$, so these formulas hold at prime-power endpoints as well. Inserting (11) in each bracket proves

$$\begin{aligned} x^{-2}\sum m \le xmA_K(m)^2 &= \tfrac{r}{2}\log x + O_K(1), \\ x^2\sum *m > xA_K(m)^2m^{-3} &= \tfrac{r}{2}\log x + O_K(1), \\ D(x) &= r\log x + O_K(1). \end{aligned} \tag{37}$$

For instance, $\int_1^x v\log v\,dv = (x^2/2)\log x - x^2/4 + 1/4$, and $\int_x^\infty v^{-3}\log v\,dv = (\log x)/(2x^2) + 1/(4x^2)$. The $O_K(v)$ remainder contributes $O_K(1)$ to both terms.

For the error moment,

$$V(x) = x^{-2}\sum_{m\le x} m^2 A_K(m)^2 + x^2\sum_{m>x} A_K(m)^2 m^{-2},$$
$$\sum_{m>x} A_K(m)^2 m^{-2} = -x^{-2}B_K(x) + 2\int_x^\infty B_K(v)v^{-3}\,dv \ll_K \frac{\log(2x)}{x}.$$

The first part of $V(x)$ is at most $B_K(x) \ll_K x\log(2x)$. It follows that

$$V(x) \ll_K x\log(2x) < \infty. \tag{38}$$

Over $1 \le x \le 2$, the same estimates follow by boundedness; thus all stated coefficient estimates are uniform for $x \ge 1$.

Montgomery–Vaughan's mean-value theorem [31, Corollary 3, equation (1.11), p. 75] states

$$\int_0^T \left|\sum_m a_m m^{-it}\right|^2 dt = T\sum_m |a_m|^2 + O\left(\sum_m m|a_m|^2\right),$$

with absolute implied constant when the final sum is finite. For an explicit limiting justification here, apply the finite version to $P_{x,M}(t) = \sum_{m\le M} a_x(m)m^{-it}$. For each fixed x, T, the tails beyond $M > x$ are bounded uniformly in t by $nx\sum_{m>M}(\log m)m^{-3/2} \to 0$. Thus $P_{x,M} \to P_x$ in $L^2(0,T)$. The nonnegative coefficient sums converge to $D(x)$ and $V(x)$, and the theorem's constant is independent of M. Passing to the limit and using (37)–(38) proves (33).

To treat the background, write $G_K(t) = n\log(t+2) + e_K(t)$ for $t \ge 0$, where $|e_K(t)| \le C_K$ by (19). Even though e_K need not be real,

$$\left||G_K(t)|^2 - n^2\log^2(t+2)\right| \le 2nC_K\log(t+2) + C_K^2.$$

The antiderivative $y(\log^2 y - 2\log y + 2)$ gives

$$\int_0^T \log^2(t+2)\,dt = TL^2 + O(TL), \qquad \int_0^T \log(t+2)\,dt \ll TL.$$

Multiplication by x^{-2} proves (34). Equation (35) follows from (20). If $1 \le x \le T$, these estimates imply

$$\|P_x\|_2 \ll_K \sqrt{Tu}, \qquad \|B_x\|_2 \ll_K \sqrt{T}\,L/x, \qquad \|R_x\|_2 \ll_K \sqrt{x}.$$

Cauchy–Schwarz gives all three inequalities in (36). $\square$

4.6 A uniform form-factor estimate

Proposition 10 (Complete error budget)**.** *Let $T \ge 3$, $L = \log T$, and $u = \log(2x)$. Uniformly for $1 \le x \le T$,*

$$2\pi F_K(x,T) = n^2Tx^{-2}L^2 + rT\log x + O_K(\mathcal{E}(x,T)), \tag{39}$$
$$\begin{aligned}\mathcal{E}(x,T) &= Tx^{-2}L + T + xu + TL\sqrt{u}/x \\ &\quad + \sqrt{Txu} + \sqrt{T/x}\,L + x + xL^3.\end{aligned} \tag{40}$$

Proof. Expand the exact identity (17) in $L^2(0,T)$:

$$\|H_x\|_2^2 = \|P_x\|_2^2 + \|B_x\|_2^2 + \|R_x\|_2^2 - 2\,\mathrm{Re}\langle P_x, B_x\rangle_2 \\ - 2\,\mathrm{Re}\langle P_x, R_x\rangle_2 + 2\,\mathrm{Re}\langle B_x, R_x\rangle_2.$$

The two main terms come from (33) and (34). Their errors give $T + xu + Tx^{-2}L$. The residual norm gives x, and (36) gives the three square-root terms. Finally, (26) contributes xL^3, since $\log(T+2) \asymp L$ for $T \geq 3$. These are exactly all the terms in (40). □

4.7 Quantitative Fourier synthesis

Proposition 11 (Integration at fixed strict support)**.** *For $0 < b < 1$ and f as in Lemma 4, let $S_{K,f}(T)$ denote the pair sum on the left of (7), and put $Q_K(T) = nT\log T/(2\pi)$. Then*

$$\frac{S_{K,f}(T)}{Q_K(T)} = nf(0) + \frac{r}{n}\int_{-1}^{1} |a| f(a)\,da + O_{K,b,f}(\mathcal{R}_b(T)). \tag{41}$$

Replacing $Q_K(T)$ by $N_K(T)$ gives (7) and hence proves Lemma 4.

Proof. The zero sum over A_T is finite. Therefore the Fourier convention and $x^z = \exp(z\log x)$ give exactly

$$S_{K,f}(T) = \int_{-b}^{b} f(a) F_K(T^a, T)\,da.$$

Exchanging the two zero variables, and using $W(-z) = W(z)$, gives $F_K(x^{-1}, T) = F_K(x, T)$. Since f is even,

$$S_{K,f}(T) = 2\int_0^b f(a) F_K(T^a, T)\,da. \tag{42}$$

This identity applies to the complex Fourier arguments without any assumption on β beyond the critical strip.

Now insert (39) with $x = T^a = e^{aL}$ and divide by $Q_K(T)$. The contribution of the first main term is

$$2nL\int_0^b f(a)e^{-2aL}\,da = nf(0) + O_{K,f}(L^{-1}).$$

For detail, extend f by zero as already prescribed outside $[-b, b]$, set $v = 2La$, and use $|f(v/(2L)) - f(0)| \leq \|f'\|_\infty v/(2L)$ in the integral against $e^{-v}\,dv$ on $[0,\infty)$. This also includes its tail beyond $2bL$. The second main term contributes exactly

$$\frac{2r}{n}\int_0^b af(a)\,da = \frac{r}{n}\int_{-1}^{1} |a| f(a)\,da.$$

For clarity, the following table lists an upper bound for the integral of each error term in (40), after substitution and division by TL. Multiplication by $2|f(a)|/n$ is absorbed in the implied constant.

Original error term	Integrated normalized bound
$Tx^{-2}L$	$O(L^{-1})$
T	$O_b(L^{-1})$
xu	$O_b(T^{b-1})$
$TL\sqrt{u}/x$	$O(L^{-1})$
$\sqrt{Txu}$	$O_b(T^{(b-1)/2}L^{-1/2})$
$\sqrt{T/x}\,L$	$O_b(T^{-1/2})$
x	$O_b(T^{b-1}/L)$
xL^3	$O_b(T^{b-1}L^2)$

Here $u = \log 2 + aL$. For the fourth row we retain the dependence of $\sqrt{u}$ on a, using the exact change of variables

$$\int_0^b e^{-aL}\sqrt{\log 2 + aL}\,da = \frac{1}{L}\int_0^{bL} e^{-v}\sqrt{\log 2 + v}\,dv \ll L^{-1}.$$

For the other rows use $\int_0^b e^{-2aL}\,da \le 1/(2L)$, $\log 2 + aL \ll_b L$, and the upper bounds at $a = b$ for the increasing exponential factors. The fifth row follows, for example, from $\sqrt{Txu}/(TL) = T^{(a-1)/2}\sqrt{\log 2 + aL}/L$. Adding the rows proves (41).

For each fixed $b < 1$, all terms in $\mathcal{R}_b(T)$ are $O_b(L^{-1})$. This follows directly from boundedness of $L^3 e^{-(1-b)L}$, $L^{1/2}e^{-(1-b)L/2}$, and $Le^{-L/2}$ for $L \ge \log 3$. Finally (27) gives, for sufficiently large T depending on K,

$$\frac{Q_K(T)}{N_K(T)} = 1 + O_K(L^{-1}).$$

Multiplying (41) by this ratio changes its error by $O_{K,b,f}(L^{-1})$, which is already included. This proves (7). $\square$

Dependence on the test function and order of limits. More precisely, the strict-support error is bounded by $C_{K,b}(\|f\|_\infty + \|f'\|_\infty)\mathcal{R}_b(T)$, with a lower threshold for T depending only on K. The stated C_c^2 class is convenient for the later applications. In particular, at fixed $b < 1$ they are uniform on bounded subsets in the C^1 norm. The field and test function are fixed before taking $T \to \infty$. When smooth cutoffs approach endpoint support in Section 5, each cutoff is treated first at its own strict support. That argument does not require uniformity as $b \uparrow 1$. The following corollary uses stronger control of the test at the boundary to reach $b = 1$.

4.8 A smooth endpoint extension

Corollary 12 (Globally smooth endpoint tests)**.** *Let $f \in C_c^2(\mathbb{R})$ be real and even, with* $\operatorname{supp} f \subset [-1,1]$, *and put $M_j = \|f^{(j)}\|_\infty$ for $j = 0,1,2$. For the same finite complex-zero pair sum $S_{K,f}(T)$,*

$$\frac{S_{K,f}(T)}{N_K(T)} = nf(0) + \frac{r}{n}\int_{-1}^{1}|a|f(a)\,da + O_K\left(\frac{M_0 + M_1 + M_2}{\log T}\right).$$

The error constant and lower threshold for T depend only on the fixed field. Thus the formula is uniform on bounded C^2 subsets of this endpoint class. No GRH is assumed.

Proof. The pointwise estimate in Proposition 10 already holds on the closed range $1 \le x \le T$. What changes is how its error is integrated. Global C^2 regularity of the zero extension implies $f(1) = f'(1) = 0$.

Taylor's integral formula gives

$$|f(a)| \le \frac{M_2}{2}(1-a)^2 \quad (0 \le a \le 1).$$

Consequently, for every $c > 0$,

$$\int_0^1 |f(a)|e^{-cL(1-a)}\,da \le \frac{M_2}{2}\int_0^\infty y^2 e^{-cLy}\,dy = \frac{M_2}{c^3L^3}.$$

Substitute $x = T^a$ and $u = \log 2 + aL$ in (40), multiply by $|f(a)|$, and integrate over $0 \le a \le 1$. After division by TL, the eight bounds are

Original error term	Weighted integrated normalized bound
$Tx^{-2}L$	$O(M_0/L)$
T	$O(M_0/L)$
xu	$O(M_2/L^3)$
$TL\sqrt{u}/x$	$O(M_0/L)$
$\sqrt{Txu}$	$O(M_2/L^{7/2})$
$\sqrt{T/x}\,L$	$O(M_0/(\sqrt{T}\,L))$
x	$O(M_2/L^4)$
xL^3	$O(M_2/L)$

For the third, fifth, seventh, and eighth rows, use the displayed boundary estimate with $c = 1$ or $1/2$ and $u \ll L$. For the fourth row use $s = aL$ and

$$\int_0^1 |f(a)|\sqrt{\log 2 + aL}\,e^{-aL}\,da \le \frac{M_0}{L}\int_0^\infty \sqrt{\log 2 + s}\,e^{-s}\,ds.$$

The first, second, and sixth rows follow by direct integration.

The finite Fourier identity (42) holds with upper endpoint 1. Its prime main term is unchanged. Since $f'(0) = 0$, Taylor's theorem at zero, using the global zero extension, gives the slightly sharper spike estimate

$$2nL\int_0^1 f(a)e^{-2La}\,da = nf(0) + O_K(M_2/L^2).$$

All eight errors are $O_K((M_0 + M_2)/L)$. Finally, $Q_K(T)/N_K(T) = 1 + O_K(1/L)$ changes the main term by $O_K(M_0/L)$ and multiplies the existing error by a bounded factor. This proves the assertion and its stated uniformity, without requiring any bound on M_2 in terms of M_0. $\square$

This endpoint extension depends on quadratic boundary vanishing; it does not cover the nonsmooth optimizer used in Section 5. For a smooth positive density p on $[-1/2, 1/2]$ with $p(1/2) > 0$, extended by zero, its convolution satisfies

$$(p * p)(1-\varepsilon) = p(1/2)^2\varepsilon + O_p(\varepsilon^2) \qquad (\varepsilon \downarrow 0).$$

Indeed, the convolution integral at this point is $\int_0^\varepsilon p(1/2-v)p(1/2-\varepsilon+v)\,dv$; Taylor expansion on the interval gives the displayed estimate and the one-sided derivative $(p*p)'(1-) = -p(1/2)^2$. It is therefore not globally C^1 at that endpoint, and its second distributional derivative has endpoint atoms. The later optimization continues to use fixed smooth cutoffs followed by a separate cutoff limit. No support beyond $[-1, 1]$, or energy asymptotic for the nonsmooth endpoint profile, is asserted here.

5 Weight removal and the exact scalar optimum

We first convert the weighted pair formula into the energy required by Lemma 3, then minimize its limiting scalar functional. Finally, fixed smooth cutoffs transfer that minimum to the zero counts. Throughout, K is fixed, $n \geq 1$ and $1 \leq r \leq n$ have the meanings in Section 2, $T \geq 3$, $L = \log T$, and $I = [-1/2, 1/2]$.

5.1 Exact weight removal and the limiting energy

Let $\eta \in C_c^\infty((-1/2, 1/2))$ be real and even, normalized by $\int_I \eta(x)^2\,dx = 1$. Extend it by zero to $\mathbb{R}$, and set

$$p = \eta^2, \qquad q = p * p, \qquad (p * p)(u) = \int_{\mathbb{R}} p(v)p(u-v)\,dv.$$

Choose a with $0 < a < 1/2$ and $\operatorname{supp} p \subset [-a, a]$. Thus q and q'' are fixed, real, even C_c^∞ functions supported in $[-2a, 2a] \subset (-1, 1)$; both satisfy Lemma 4. With $z = i(\rho - \rho')L/(2\pi)$, Fourier differentiation gives exactly

$$\left(\widehat{q}(z) - \frac{\widehat{q''}(z)}{4L^2}\right) W(\rho - \rho') = \widehat{p}(z)^2. \tag{43}$$

Indeed, compact support gives $\widehat{q} = (\widehat{p})^2$ at every complex argument. Integration by parts twice, with no boundary terms, gives $\widehat{q''}(z) = -4\pi^2 z^2 \widehat{q}(z)$, and $\pi^2 z^2/L^2 = -(\rho - \rho')^2/4$. The multiplier on the left therefore cancels W exactly, also on the diagonal $\rho = \rho'$. This is Lamzouri's differential weight-removal device [1, Section 3, proof of Lemma 3.2].

Using the distinct zero set and multiplicities of Section 2, define the unweighted energy by

$$E_K(\eta, T) = \sum_{\rho, \rho' \in \mathcal{Z}_K(T)} m_\rho m_{\rho'} \widehat{p}\left(i(\rho - \rho')\frac{L}{2\pi}\right)^2.$$

The square is an ordinary square, not an absolute square. The sum is over ordered pairs, including the diagonal. If $S_{K,g}(T)$ denotes the weighted pair sum from Section 4, then (43) yields the finite-sum identity

$$E_K(\eta, T) = S_{K,q}(T) - \frac{S_{K,q''}(T)}{4L^2}.$$

Apply Lemma 4 separately to the two fixed tests. In particular $S_{K,q''}(T)/N_K(T) = O_{K,\eta}(1)$, so its contribution above is $O_{K,\eta}(L^{-2})$. No uniformity in a moving test is required. Evenness of p gives

$$q(0) = \int_I p(x)^2\,dx, \qquad \int_{-1}^{1} |u| q(u)\,du = \iint_{I^2} |x + y| p(x)p(y)\,dx\,dy = \iint_{I^2} |x - y| p(x)p(y)\,dx\,dy.$$

The first double integral follows from Fubini and the second from $y \mapsto -y$. Consequently, for each fixed η,

$$\frac{E_K(\eta, T)}{N_K(T)} = J_{n,r}(p) + o(1), \quad J_{n,r}(p) = n\int_I p^2 + \frac{r}{n}\iint_{I^2} |x - y| p(x)p(y)\,dx\,dy,$$

as $T \to \infty$. In fact Section 4 gives an $O_{K,\eta}(L^{-1})$ error here. Positivity of the total energy will follow from its identification with Lemma 3 below; no pointwise positivity of the complex summands is being used.

5.2 The scalar minimizer and a strict variational certificate

For this variational step, extend $J_{n,r}$ to the real affine space

$$\mathcal{A} = \{p \in L^2(I;\mathbb{R}) : \int_I p(x)\,dx = 1\}.$$

The double integral is absolutely convergent because $|x-y| \le 1$ on I^2 and $L^2(I) \subset L^1(I)$. Members of $\mathcal{A}$ need not be even or nonnegative; this enlargement is only for the minimization, not an assertion that the pair formula applies to every member. Put

$$\omega = \frac{\sqrt{2r}}{n}, \qquad p_*(x) = \frac{\omega\cos(\omega x)}{2\sin(\omega/2)}, \quad x \in I.$$

Since $0 < \omega/2 \le 1/\sqrt{2} < \pi/2$, the density p_* is positive on the closed interval and is smooth and even there. Direct integration gives $\int_I p_* = 1$. Define its potential using

$$\mathcal{E}_p(x) = np(x) + \frac{r}{n}\int_I |x-y|p(y)\,dy.$$

The identity $\partial_x^2|x-y| = 2\delta_y$, or differentiation of the two integrals on either side of $y = x$, gives $\mathcal{E}_{p_*}'' = np_*'' + (2r/n)p_* = 0$ in the interior of I. Thus $\mathcal{E}_{p_*}$ is affine, and evenness makes it constant. Continuity extends that constant to the endpoints. Since $\int_I yp_*(y)\,dy = 0$, evaluation at $x = 1/2$ gives

$$\mathcal{E}_{p_*}(1/2) = np_*(1/2) + \frac{r}{2n} = \frac{r}{2n} + \sqrt{\frac{r}{2}}\cot\left(\frac{1}{n}\sqrt{\frac{r}{2}}\right) = C_{n,r}.$$

Multiplication by p_* and integration show $J_{n,r}(p_*) = C_{n,r}$.

Let $h \in L^2(I;\mathbb{R})$ with $\int_I h = 0$, and set

$$H(x) = \int_{-1/2}^{x} h(y)\,dy, \qquad U_h(x) = \int_I |x-y|h(y)\,dy.$$

Then $H \in H_0^1(I)$: it is absolutely continuous, has weak derivative $h \in L^2$, and vanishes at both endpoints. Differentiating U_h gives $U_h'(x) = 2H(x)$ almost everywhere. Hence integration by parts, whose boundary term vanishes because H does, gives

$$\iint_{I^2} |x-y|h(x)h(y)\,dx\,dy = \int_I H'(x)U_h(x)\,dx = -2\int_I H(x)^2\,dx.$$

The linear term in $J_{n,r}(p_* + h)$ is $2\int_I \mathcal{E}_{p_*}(x)h(x)\,dx = 0$. The Dirichlet Poincaré inequality on the interval of length one is $\int_I H^2 \le \pi^{-2}\int_I |H'|^2$. For example, expansion in the orthonormal sine basis $\sqrt{2}\sin(m\pi(x+1/2))$, $m \ge 1$, gives $\|H'\|_2^2 = \sum_{m\ge1}(m\pi)^2 a_m^2 \ge \pi^2\sum_{m\ge1} a_m^2 = \pi^2\|H\|_2^2$, where a_m are the expansion coefficients. Combining these facts yields

$$J_{n,r}(p_* + h) - C_{n,r} = n\int_I h^2 - \frac{2r}{n}\int_I H^2 \ge \left(n - \frac{2r}{n\pi^2}\right)\int_I h^2. \tag{44}$$

Since $r \le n$, the coefficient is at least $n - 2/\pi^2 > 0$. Thus p_* is the unique minimizer on $\mathcal{A}$, up to equality almost everywhere, even among signed, not necessarily even functions.

This verifies a known generalized Montgomery–Taylor constant, not a new extremal value. Specifically, Theorem 5 of [28], with $c_1 = n$, $c_2 = r/n$ and $\Delta = 1$, gives $C_{n,r} = K_\nu(0,0)^{-1}$. Here K_ν is the reproducing kernel of their weighted Paley–Wiener space for the measure $d\nu = n\delta_0 + (r/n)|u|\,du$

supported in $[-1,1]$. The parameter condition holds because $(c_2/c_1)\Delta^2 = r/n^2 \le 1 < 5/3$. The minimum proved here is for $J_{n,r}$ on $\mathcal{A}$, not for every larger Fourier-test class or every refinement of the finite certificate.

For later use, Cauchy–Schwarz gives $\int_I p_*^2 \ge 1$, and the distance integral of the positive density p_* is strictly positive. Thus $C_{n,r} > n$. The uniform density $p = 1$ on I satisfies

$$\iint_{I^2} |x-y|\,dx\,dy = 2\int_{-1/2}^{1/2}\int_{-1/2}^{x} (x-y)\,dy\,dx = \frac{1}{3}, \qquad J_{n,r}(1) = n + \frac{r}{3n}.$$

It differs from p_* because $\omega > 0$, so (44) gives

$$n < C_{n,r} < n + \frac{r}{3n} \le n + \frac{1}{3}.$$

In particular $\lfloor C_{n,r} \rfloor = n$. By the threshold comparison in Section 3, $k = n$ is the unique maximizing integer for the basic bound $(2k+1-C_{n,r})/[k(k+1)]$. This does not assert that the same k optimizes bounds containing additional simple-zero or nonreal-zero information.

5.3 Smooth cutoffs and the distinct-zero consequence

The zero extension of p_* is not a smooth compactly supported test, so we do not apply Lemma 4 to it or to its endpoint convolution. For integers $j \ge 5$, choose real even $\chi_j \in C_c^\infty((-1/2+1/j, 1/2-1/j))$ such that $0 \le \chi_j \le 1$ and $\chi_j = 1$ on $|x| \le 1/2 - 2/j$. Define

$$c_j = \int_I \chi_j^2 p_*, \qquad \eta_j = \frac{\chi_j \sqrt{p_*}}{\sqrt{c_j}}, \qquad p_j = \eta_j^2 = \frac{\chi_j^2 p_*}{c_j},$$

with zero extensions outside I. Positivity and smoothness of p_* give $c_j > 0$ and real even $\eta_j \in C_c^\infty((-1/2, 1/2))$ with $\int_I \eta_j^2 = 1$. Dominated convergence gives $c_j \to 1$ and $p_j \to p_*$ in both $L^1(I)$ and $L^2(I)$.

For the distance form $\mathcal{D}(p) = \iint_{I^2} |x-y| p(x)p(y)\,dx\,dy$, the bound

$$|\mathcal{D}(p) - \mathcal{D}(v)| \le \|p - v\| * L^1(I)(\|p\| * L^1(I) + \|v\|_{L^1(I)})$$

follows by expanding the difference and using $|x-y| \le 1$. Together with L^2 convergence, it proves $C_j := J_{n,r}(p_j) \to J_{n,r}(p_*) = C_{n,r}$.

For each sufficiently large T, map the zero multiset into $\mathbb{C}$ by

$$z_\rho = i(\rho - 1/2)\frac{L}{2\pi}.$$

This map is injective, preserves multiplicities, and satisfies $z_{1-\bar\rho} = \overline{z_\rho}$. Reflection preserves the window $0 < \gamma \le T$, so the transformed multiset satisfies the hypotheses of Lemma 3. Its N and D are exactly $N_K(T)$ and $D_K(T)$, and its energy, with $K_{\eta_j} = \widehat{p}_j$, is exactly $E_K(\eta_j, T)$. For the fixed integer $k = n$, that lemma gives

$$\frac{D_K(T)}{N_K(T)} \ge \frac{2n+1-E_K(\eta_j,T)/N_K(T)}{n(n+1)}.$$

For every fixed j, the energy limit proved above therefore yields

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \ge \frac{2n+1-C_j}{n(n+1)}.$$

Only now let $j \to \infty$. The left side is independent of j, so this proves (1). No estimate uniform in the cutoff index is used, and $k = n$ need not be optimal for each individual C_j. Taking a cutoff family of the uniform density in the same way gives

$$\frac{2n+1-(n+r/(3n))}{n(n+1)} = \frac{1}{n} - \frac{r}{3n^2(n+1)} \geq \frac{3n+2}{3n(n+1)},$$

which proves (2). The strict comparison with $J_{n,r}(1)$ proves the asserted strict improvement in Theorem 1; its geometric increment is proved separately in Section 7.

Finally, the dependence on rank can be checked without differentiating the cotangent. For fixed n and $1 \leq r' < r'' \leq n$, let $p_{r''}$ be the positive minimizer for $J_{n,r''}$. Then

$$C_{n,r''} = J_{n,r'}(p_{r''}) + \frac{r''-r'}{n}\mathcal{D}(p_{r''}) > C_{n,r'},$$

because $\mathcal{D}(p_{r''}) > 0$ and $J_{n,r'}(p_{r''}) \geq C_{n,r'}$. Hence $b_{n,r}$ decreases strictly with r, and the specialization $r = n$ gives the degree-only bounds in Table 1. These are bounds for distinct complex zero locations; this argument does not replace D_K by a count of simple or critical-line zeros.

6 Abelian critical-line zeros and the simplicity distinction

This section separates three assertions: the known simple-critical-line bound for each fixed primitive Dirichlet factor, its multiplicity-weighted consequence for an abelian Dedekind zeta function, and a Galois obstruction to transferring simplicity to the product. All fields and characters are fixed before $T \to \infty$.

6.1 The fixed primitive Dirichlet argument

For a primitive Dirichlet character χ, let $\mathcal{Z}_\chi(T)$ be the distinct zeros of $L(s,\chi)$ with $0 < \operatorname{Re}\rho < 1$ and $0 < \operatorname{Im}\rho \leq T$, and let $m_{\chi,\rho}$ be their analytic multiplicities. Define

$$N_\chi(T) = \sum_{\rho\in\mathcal{Z}_\chi(T)} m_{\chi,\rho}, \qquad N_{\chi,0}(T) = \sum_{\substack{\rho\in\mathcal{Z}_\chi(T)\\ \operatorname{Re}\rho=1/2}} m_{\chi,\rho},$$

$$S_\chi(T) = \bigl|\{\rho \in \mathcal{Z}_\chi(T) : \operatorname{Re}\rho = 1/2,\ m_{\chi,\rho} = 1\}\bigr|.$$

In particular $S_\chi(T) \leq N_{\chi,0}(T)$. Real zeros are excluded from these counts. The factorwise result

$$\liminf_{T\to\infty} \frac{S_\chi(T)}{N_\chi(T)} \geq 2 - C_{1,1} = c_0$$

is already contained in [14, Theorems A–B], including the positive-height window used here. We explain its separate derivation from Lamzouri's certificate and the analytic argument of Section 4. This is not a direct substitution into Lemma 4, whose statement concerns Dedekind zeta functions.

The unique principal primitive character has conductor one and gives $\zeta(s)$, already covered. Henceforth in this subsection let χ be nonprincipal, of conductor Q_χ, and write $\chi(-1) = (-1)^\varepsilon$ with $\varepsilon \in \{0,1\}$. Set

$$Z_\chi(s) = \frac{L'(s,\chi)}{L(s,\chi)}, \qquad \Lambda_\chi(s) = (Q_\chi/\pi)^{(s+\varepsilon)/2}\Gamma((s+\varepsilon)/2)L(s,\chi).$$

We use the classical analytic properties of primitive Dirichlet L-functions: Λ_χ is entire of order one, has precisely the nontrivial zeros in $0 < \operatorname{Re} s < 1$, and is nonzero at 0 and 1. The functional equation and conjugation identity are

$$\Lambda_\chi(s) = w_\chi \Lambda_{\bar\chi}(1-s) = w_\chi \overline{\Lambda_\chi(1-\bar s)}, \qquad |w_\chi| = 1.$$

For the Euler product, functional equation and nonvanishing at 1, see the NIST Digital Library of Mathematical Functions, Section 25.15, especially equations (25.15.2), (25.15.5) and (25.15.9). The nonvanishing of the completion at 0 follows from that at 1 and the functional equation. For a nonreal χ, neither $\Lambda_\chi(\bar s) = \overline{\Lambda_\chi(s)}$ nor symmetry of its zeros about the real axis is assumed.

Local counts and the correct reflection. Put $A_\chi = \Lambda'_\chi/\Lambda_\chi$. Differentiating the last functional equation gives

$$A_\chi(s) = -\overline{A_\chi(1-\bar s)}.$$

Thus $\rho \mapsto 1-\bar\rho$ preserves the zeros of this same factor, their multiplicities, and their imaginary parts. The Hadamard argument in Lemma 5 uses only this symmetry of real parts, not a real-valued Dirichlet series. Indeed its real-part constant vanishes by adding the formulas at $s = 2$ and $s = -1$. Consequently

$$\begin{aligned} U_\chi(t) &:= \sum_\rho \frac{2-\operatorname{Re}\rho}{(2-\operatorname{Re}\rho)^2 + (t-\operatorname{Im}\rho)^2} = \operatorname{Re} A_\chi(2+it) \\ &= \frac{1}{2}\log(|t|+2) + O_\chi(1), \qquad \sum_\rho \frac{1}{1+(t-\operatorname{Im}\rho)^2} \ll_\chi \log(|t|+2). \end{aligned}$$

Here and in the unrestricted sums below, all nontrivial zeros of $L(s,\chi)$ are counted with multiplicity, including real zeros. Absolute convergence follows as in that lemma from order one and the critical strip. The last equality uses $|Z_\chi(2+it)| \le \sum_m \Lambda(m)m^{-2}$ and Stirling's estimate. Integrating this Lorentz kernel over $0 \le t \le T$, with the same boundary-mass estimate as in (32), gives

$$\int_0^T U_\chi(t)\,dt = \pi N_\chi(T) + O_\chi(\log^2(T+2)), \qquad N_\chi(T) = \frac{T\log T}{2\pi} + O_\chi(T).$$

This is a positive-height normalization; it does not follow by halving a two-sided count for a nonreal character.

Exact resolvent and arithmetic input. Logarithmic differentiation before reflection gives

$$-Z_\chi(s) = Z_{\bar\chi}(1-s) + \log(Q_\chi/\pi) + \frac{1}{2}\psi((s+\varepsilon)/2) + \frac{1}{2}\psi((1-s+\varepsilon)/2), \qquad \psi = \Gamma'/\Gamma.$$

In particular $G_\chi(t) := -Z_\chi(-1/2+it)$ is the exact background

$$\begin{aligned} G_\chi(t) &= Z_{\bar\chi}(3/2-it) + \log(Q_\chi/\pi) \\ &\quad + \frac{1}{2}\psi\Big(\frac{\varepsilon - 1/2 + it}{2}\Big) + \frac{1}{2}\psi\Big(\frac{\varepsilon + 3/2 - it}{2}\Big) = \log(|t|+2) + O_\chi(1). \end{aligned}$$

The bounded error may be complex. The reflected logarithmic derivative is $Z_{\bar\chi}$, not Z_χ. The two gamma arguments stay away from poles, including for bounded t; Stirling gives the displayed asymptotic for large $|t|$.

For $x \geq 1$ and $t \in \mathbb{R}$ define, with real logarithms for positive bases,

$$H_{\chi,x}(t) = 2 \sum_{\rho} \frac{x^{\rho-1/2-it}}{1-(\rho-1/2-it)^2},$$
$$P_{\chi,x}(t) = \sum_{m \geq 2} \Lambda(m)\chi(m)m^{-1/2-it} \min(m/x, x/m).$$

The rational Mellin calculation of Proposition 7 then gives the exact identity

$$H_{\chi,x}(t) = -P_{\chi,x}(t) + x^{-1}G_\chi(t) + R_{\chi,x}(t),$$
$$R_{\chi,x}(t) = 2 \sum_{j \geq 0} \frac{b_{\chi,j}x^{-j-1/2-it}}{(j-1/2+it)(j+3/2+it)}, \qquad b_{\chi,j} = \mathbf{1}_{j \equiv \varepsilon \pmod 2}.$$

There is no pole-at-one residue. The trivial zeros at $-\varepsilon - 2j$, $j \geq 0$, are simple: the completion is nonzero there and the corresponding gamma pole is simple. In particular an even nonprincipal character has a simple trivial zero at 0. The contour limits are justified by the same order-one bounds on the horizontal sides and by $Z_\chi(-M-1/2+iu) \ll_\chi \log(M+|u|+2)$ on the left sides, where $M \geq 1$ is an integer; digamma reflection is valid for both parities, whose shifted arguments are quarter-integers away from poles. All three series converge absolutely and locally uniformly, also at $x = 1$. The denominator estimate used to prove (20) gives the stronger bounds

$$R_{\chi,x}(t) \ll_\chi \frac{x^{-1/2}}{|t|+2}, \qquad \int_{\mathbb{R}} |R_{\chi,x}(t)|^2 \, dt \ll_\chi x^{-1} \leq x.$$

For $y \geq 2$, the arithmetic diagonal uses modulus squares:

$$\sum_{m \leq y} |\Lambda(m)\chi(m)|^2 = y \log y + O_\chi(y),$$

by the prime number theorem and partial summation; higher prime powers contribute $O(\sqrt{y}\log^3(2y)) = O(y)$. Removing all powers of the finitely many primes dividing Q_χ costs $O_\chi(\log(2y))$. Positivity of the original coefficients is unnecessary: $|\Lambda(m)\chi(m)| \leq \Lambda(m)$ supplies every absolute bound, and Montgomery–Vaughan [31, Corollary 3] applies to complex coefficients through their modulus squares. Thus, for $x \geq 1$ and $T \geq 3$, with norms on $[0, T]$,

$$\|P_{\chi,x}\| * 2^2 = T \log x + O * \chi(T + x\log(2x)),$$
$$\|x^{-1}G_\chi\| * 2^2 = Tx^{-2}(\log T)^2 + O * \chi(Tx^{-2}\log T).$$

Fixed-test synthesis and the finite certificate. Let $H_{\chi,x,T}$ be the finite sum defining $H_{\chi,x}$ restricted to $0 < \operatorname{Im}\rho \leq T$, now defined for every $x > 0$, and put

$$F_\chi(x,T) = \sum_{\rho,\rho' \in \mathcal{Z}_\chi(T)} m_{\chi,\rho}m_{\chi,\rho'}x^{\rho-\rho'}W(\rho-\rho').$$

The internal reflection preserves this window, so the finite Plancherel calculation of Proposition 8 gives $F_\chi(x,T) = (2\pi)^{-1}\int_{\mathbb{R}} |H_{\chi,x,T}(t)|^2 dt$. Its localization uses the Lorentz bound above and gives an $O_\chi(x\log^3(T+2))$ error. The two mean squares, the residual bound and Cauchy–Schwarz give every term of (40) with $n = r = 1$ and a constant depending on χ. Exchanging the two zero variables gives

$F_\chi(x^{-1},T) = F_\chi(x,T)$ without invoking symmetry about the real axis. The fixed-support integration of Proposition 11 therefore proves, for fixed $0 < b < 1$ and real even $f \in C_c^2(\mathbb{R})$ supported in $[-b,b]$,

$$\sum_{\rho,\rho' \in \mathcal{Z}_\chi(T)} m_{\chi,\rho} m_{\chi,\rho'} \widehat{f}\left(i(\rho-\rho')\frac{\log T}{2\pi}\right) W(\rho-\rho')$$
$$= N_\chi(T)\left(f(0) + \int_{-1}^{1} |u| f(u)\,du + O_{\chi,b,f}(1/\log T)\right).$$

All pairs are ordered, including the diagonal. The height T may be a zero ordinate. No conductor-aspect uniformity is claimed.

Apply this formula separately to $q_j = p_j * p_j$ and q_j'', where $p_j = \eta_j^2$ is a fixed smooth cutoff from Section 5 with $n = r = 1$. Weight removal gives the ordinary-square energy $E_\chi(\eta_j, T)$, defined as E_K there with the zero set and multiplicities of $L(s,\chi)$, and

$$E_\chi(\eta_j, T)/N_\chi(T) = J_{1,1}(p_j) + o_{\chi,j}(1).$$

The map $\rho \mapsto i(\rho - 1/2)\log T/(2\pi)$ is injective, preserves multiplicities, and sends the internal reflection to complex conjugation. Its real simple points are exactly the zeros counted by $S_\chi(T)$. Lamzouri's inequality [1, Proposition 2.1] consequently gives $S_\chi(T) \geq 2N_\chi(T) - E_\chi(\eta_j, T)$. First take $\liminf_{T\to\infty}$ with j fixed, then let $j \to \infty$ to obtain $\liminf S_\chi/N_\chi \geq 2 - C_{1,1} = c_0$. This recovers the cited primitive Dirichlet result; it does not constitute a new factorwise percentage.

6.2 What the abelian factorization preserves

For abelian $K/\mathbb{Q}$, class field theory gives the exact factorization

$$\zeta_K(s) = \prod_{\chi \in X_K} L(s,\chi), \qquad |X_K| = n.$$

Here X_K denotes the primitive Dirichlet characters associated to the characters of $\mathrm{Gal}(K/\mathbb{Q})$, including the trivial character of conductor one. Each factor is taken with its own primitive conductor; there are no omitted Euler factors at ramified primes. See [18, Section 1] for the abelian factorization and [26, Lemma 2.2 and Corollary 2.3] for the Artin-factor formalism. All factors are holomorphic in the open critical strip. At every point there, their nonnegative orders add, including at common zeros. Hence

$$N_{K,0}(T) = \sum_{\chi \in X_K} N_{\chi,0}(T), \qquad N_K(T) = \sum_{\chi \in X_K} N_\chi(T).$$

For any $\delta > 0$, each fixed factor satisfies $N_{\chi,0}(T) \geq S_\chi(T) \geq (c_0 - \delta)N_\chi(T)$ for all sufficiently large T. There are only finitely many factors, so a single threshold works for all of them. Sum, divide by $N_K(T)$, and let $\delta \downarrow 0$ to prove Corollary 2. Equivalently, the fixed-factor asymptotics $N_\chi(T) = T\log T/(2\pi) + O_K(T)$ show that the finitely many individual errors remain $o(N_K(T))$ after summation.

These are multiplicity identities, not identities for simple or distinct zero counts. Locally, two holomorphic factors with expansions $a(s-\rho) + O((s-\rho)^2)$ and $b(s-\rho) + O((s-\rho)^2)$, $ab \neq 0$, have a product with a double zero. This illustrates the obstruction; it does not assert a common zero for a particular pair of Dirichlet factors. No common-zero estimate is used in the proof.

6.3 Limits on transferring simplicity to the product

For $K \neq \mathbb{Q}$, the scalar minimum satisfies $C_{n,r} > n \geq 2$. Consequently the energy estimate from Section 5 combined with $S \geq 2N - E$ yields only the negative limiting lower bound $2 - C_{n,r}$, and hence no positive simple-critical proportion. This describes that specific certificate and scalar test class, not the true statistic or every possible refinement.

There is also a structural upper bound for simplicity anywhere in the strip. Suppose $K/\mathbb{Q}$ is Galois with group G. Write $G^{\text{ab}} = G/[G,G]$, where $[G,G]$ is the commutator subgroup, and let $\widehat{G}$ denote the irreducible complex characters of G. Then

$$\limsup_{T\to\infty} \frac{N_K^{\text{simple}}(T)}{N_K(T)} \leq \frac{|G^{\text{ab}}|}{|G|} = \frac{1}{|[G,G]|}. \tag{45}$$

Indeed, fix a simple nontrivial zero ρ of ζ_K. Stark's local theorem [27, Theorem 3], in the form [26, Lemma 3.1], makes every Artin factor $L(s,\theta,K/\mathbb{Q})$ holomorphic at ρ. The theorem applies because $\rho \neq 1$ and $\operatorname{ord}_\rho \zeta_K = 1$. Thus $a_\theta := \operatorname{ord}_\rho L(s,\theta,K/\mathbb{Q})$ is a nonnegative integer. The meromorphic Artin factorization $\zeta_K(s) = \prod_{\theta\in\widehat{G}} L(s,\theta,K/\mathbb{Q})^{\theta(1)}$ from [26, Corollary 2.3] gives

$$1 = \operatorname{ord}_\rho \zeta_K(s) = \sum_{\theta\in\widehat{G}} \theta(1) a_\theta.$$

The summands are nonnegative integers, so precisely one has value one: its character has degree one and its local order is one; all other local orders vanish. In particular there is no cancellation by poles at this zero, even though global Artin holomorphy is not assumed.

The degree-one characters are exactly the characters of the finite abelian group G^{ab}, and there are $|G^{\text{ab}}|$ of them. Over $\mathbb{Q}$ their Artin L-functions are primitive Dirichlet L-functions, with the trivial character giving ζ. Assigning each simple Dedekind zero to its unique vanishing degree-one factor therefore gives the finite inequality

$$N_K^{\text{simple}}(T) \leq \sum_{\substack{\theta\in\widehat{G} \\ \theta(1)=1}} N_\theta(T),$$

where N_θ counts that Dirichlet factor's zeros in the same positive-height window, with multiplicity. The right side is $|G^{\text{ab}}| T \log T/(2\pi) + O_K(T)$, whereas $N_K(T) = |G| T \log T/(2\pi) + O_K(T)$. Division and a limit superior prove (45). This is a consequence of Stark's local theorem and standard zero counting, not a new holomorphy theorem.

For $G = S_3$, one has $[G,G] = A_3$ of order three and $|G^{\text{ab}}| = 2$, so the ceiling is $1/3$. This example concerns the Galois field of degree six, not a non-Galois cubic subfield. Such a field exists: the splitting field of $x^3 - 2$ has group S_3, since the polynomial is Eisenstein at 2 and its discriminant -108 is not a rational square. Indeed the transitive cubic groups are A_3 and S_3, and a nonsquare discriminant excludes A_3. Thus a universal 67.25% simple-zero transfer is impossible. For an abelian group the ceiling is merely one, which supplies no additional positive lower bound or assertion of product simplicity.

7 A strict geometric refinement without GRH

Throughout this section, K is a fixed number field of degree $n \geq 2$, and $r = |H\backslash G/H|$ has the meaning fixed in Section 2; in particular $1 \leq r \leq n$. The counts $D_K(T)$ and $N_K(T)$ concern distinct complex zero locations and zeros with multiplicity, respectively. The refinement below is therefore a distinct-zero bound, not a simple-zero or critical-line proportion. Its additional input is the bounded interval occupied by the real elements of the normalized zero multiset.

Theorem 13. *Let $n \geq 2$, $C = C_{n,r}$ and $b_0 = (2n+1-C)/(n(n+1))$. Put*

$$\lambda = \frac{\sqrt{r/2}}{n}, \qquad c = \frac{\lambda \tan \lambda}{\pi},$$

$$\delta = \min\left\{ \left(\frac{\lambda^2}{\pi^2 - \lambda^2}\right)^2, \frac{6\lambda^4}{81\pi^4(1+c^2)^3} \right\}, \qquad q_0 = \frac{b_0}{3} - \frac{2}{9n},$$

$$a = 2\delta q_0, \qquad B = (2n+6)\sqrt{C}.$$

Then $q_0 > 0$ and

$$\boxed{\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq b_0 + \epsilon_{n,r}, \quad \epsilon_{n,r} = \frac{1}{n(n+1)}\left(\frac{2a}{\sqrt{B^2+4a}+B}\right)^2 > 0.} \tag{46}$$

For every fixed quadratic field, in particular,

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq \frac{9/2 - \cot(1/2)}{6} + 10^{-15}. \tag{47}$$

Proof. Step 1: an explicit three-point kernel bound. Write

$$p_\lambda(u) = \frac{\lambda}{\sin \lambda} \cos(2\lambda u) \mathbf{1}_{[-1/2,1/2]}(u), \qquad K_\lambda = \widehat{p_\lambda}.$$

Since $1 \leq r \leq n$ and $n \geq 2$, we have $0 < \lambda \leq 1/2$. Thus p_λ is nonnegative, even, and has integral one; in particular K_λ is real on $\mathbb{R}$ and $K_\lambda(0) = 1$. For any $R > 2$, set

$$\delta_R = \min\left\{ \left(\frac{\lambda^2}{\pi^2 - \lambda^2}\right)^2, \frac{6\lambda^4}{\pi^4 R^4 (1+c^2)^3} \right\}.$$

This constant is positive and less than one. Order any three real locations of diameter at most R and let u, v be their consecutive gaps. Then

$$K_\lambda(u)^2 + K_\lambda(v)^2 + K_\lambda(u+v)^2 \geq \delta_R, \qquad u, v \geq 0, \quad u + v \leq R. \tag{48}$$

Direct integration gives the nonsingular formula

$$K_\lambda(t) = \frac{\lambda}{2\sin\lambda}\left\{ \frac{\sin(\lambda - \pi t)}{\lambda - \pi t} + \frac{\sin(\lambda + \pi t)}{\lambda + \pi t} \right\},$$

where $\sin v/v$ is assigned the value one at $v = 0$. Equivalently,

$$K_\lambda(t) = \frac{\lambda^2 \cos(\pi t) - \pi t \lambda \cot\lambda \sin(\pi t)}{\lambda^2 - \pi^2 t^2},$$

with removable singularities at $t = \pm\lambda/\pi$. The integral representation gives, for $0 < t < 1$,

$$K'_\lambda(t) = -4\pi \int_0^{1/2} u p_\lambda(u) \sin(2\pi t u)\, du < 0.$$

Hence K_λ decreases on $[0,1]$ and $K_\lambda(1) = \lambda^2/(\pi^2-\lambda^2) > 0$. If $u \le 1$ or $v \le 1$, the first term in δ_R suffices. Otherwise $u, v > 1$. Put $s = u+v$, $A = u^2+uv+v^2$, and $z_t = e^{i\pi t}(t-ic)$. The exponential phases cancel, and multiplication gives the exact identity

$$z_u z_v \bar{z}_s = (u-ic)(v-ic)(s+ic) = uvs - ic(A+c^2).$$

For unit complex numbers α, β, $|\operatorname{Im}(\alpha\beta)| \le |\operatorname{Im}\alpha| + |\operatorname{Im}\beta|$; iterating gives the corresponding three-factor inequality. Furthermore, for $t > 1$,

$$\frac{|\operatorname{Im} z_t|}{|z_t|} = \frac{\pi^2 t^2 - \lambda^2}{\pi\lambda\cot\lambda\sqrt{t^2+c^2}}|K_\lambda(t)| \le \frac{\pi t}{\lambda\cot\lambda}|K_\lambda(t)|, \qquad |z_t| \le t\sqrt{1+c^2}.$$

Apply the three-factor inequality to $z_u/|z_u|$, $z_v/|z_v|$, and $\bar{z}_s/|z_s|$. Cauchy–Schwarz and $u^2+v^2+s^2 = 2A$ then give

$$\frac{c(A+c^2)}{uvs(1+c^2)^{3/2}} \le \frac{\pi}{\lambda\cot\lambda}\left(u|K_\lambda(u)| + v|K_\lambda(v)| + s|K_\lambda(s)|\right) \le \frac{\pi\sqrt{2A}}{\lambda\cot\lambda}\left(\sum_{t=u,v,s} K_\lambda(t)^2\right)^{1/2}.$$

Squaring, using $c\lambda\cot\lambda = \lambda^2/\pi$ and $(A+c^2)^2/A \ge A$, yields

$$\sum_{t=u,v,s} K_\lambda(t)^2 \ge \frac{\lambda^4}{2\pi^4(1+c^2)^3}\frac{u^2+uv+v^2}{u^2v^2(u+v)^2} \ge \frac{6\lambda^4}{\pi^4 R^4(1+c^2)^3}.$$

For the last inequality put $w = uv \le s^2/4$; the numerator $s^2 - w$ is at least $3s^2/4$, while $w^2s^2 \le s^6/16$. Hence $(s^2-w)/(w^2s^2) \ge 12/s^4 \ge 12/R^4$. This proves (48) without numerical minimization.

Step 2: projection away from the nonreal vectors. We now make a finite argument for an arbitrary real, even, compactly supported $\eta \in L^2(\mathbb{R})$ with $\int \eta^2 = 1$. Use K_η and a nonempty finite conjugation-invariant multiset Z with support Ω and positive integer multiplicities m_z, where $m_{\bar{z}} = m_z$. Retain N, D, N_C and the real Hilbert space $\mathcal{H}_0$ from Lemma 3. The kernel in this step need not be K_λ; we will apply the argument to smooth approximations in Step 5. With the functions of Lamzouri [1, proof of Proposition 2.1], the real symmetric finite-rank operator from that lemma decomposes as

$$F = F_R + F_C, \quad F_R = \sum_{x\in\Omega\cap\mathbb{R}} m_x f_x \otimes f_x, \quad F_C = 2\sum_{z\in\mathcal{P}} m_z(g_z\otimes g_z - h_z\otimes h_z),$$

where $\mathcal{P}$ contains one representative of each nonreal conjugate pair, and $E = \|F\|_{\mathrm{HS}}^2$. Let N_C be the nonreal mass, $c_1 = |\mathcal{P}|$, and D_R the number of distinct real locations. Thus $D = D_R + 2c_1$, $N_C = 2\sum_{z\in\mathcal{P}} m_z$, and $\operatorname{tr} F_C = N_C$, since $\|g_z\|^2 - \|h_z\|^2 = 1$. Let Π project orthogonally onto the real span of all g_z, h_z with $z \in \mathcal{P}$, and set $P = I - \Pi$. The range and co-range of F_C lie in this span, so $\Pi F_C = F_C\Pi = F_C$ and $PF_CP = 0$. The lost real mass $\mathcal{L} = \sum_{x\in\Omega\cap\mathbb{R}} m_x\|\Pi f_x\|^2$ satisfies

$$\mathcal{L} + N_C = \operatorname{tr}(\Pi F), \qquad \mathcal{L} \le \sqrt{2c_1E} - N_C \le \sqrt{N_CE} - N_C. \tag{49}$$

Indeed, $0 \le \operatorname{tr}(\Pi F) \le \|\Pi\|_{\mathrm{HS}}\|F\|_{\mathrm{HS}} = \sqrt{\operatorname{rank}\Pi}\sqrt{E}$ by Hilbert–Schmidt Cauchy–Schwarz, and $\operatorname{rank}\Pi \le 2c_1 \le N_C$. Compression by an orthogonal projection contracts the Hilbert–Schmidt norm, so

$$\|PF_RP\| * \mathrm{HS}^2 = \|PFP\| * \mathrm{HS}^2 \le E.$$

Neither this contraction nor the trace estimate requires F to be positive semidefinite. If there are no nonreal locations, take $\Pi = 0$; the same identities hold with $N_C = c_1 = \mathcal{L} = 0$.

Step 3: retaining disjoint triples after compression. Fix an integer threshold $h \geq 1$. For every integer $m \geq 1$ and $0 \leq t \leq 1$,

$$m^2 t^2 \geq (2h+1)m - h(h+1) - 2(h+1)m(1-t).$$

After subtraction the difference is $(mt - h - 1)^2 + m - h - 1$. It is nonnegative when $m \geq h+1$. When $m \leq h$, put $q = h + 1 - m \geq 1$; since $mt \leq m$, the difference is at least $q^2 - q \geq 0$. For a real location x, let $\ell_x = \|\Pi f_x\|^2 \in [0,1]$. Then $\|Pf_x\|^2 = 1 - \ell_x$. For two real locations, write $a_{xy} = \langle f_x, f_y \rangle = K_\eta(x-y)$ and $b_{xy} = \langle Pf_x, Pf_y \rangle$. These are real numbers of absolute value at most one, and $a_{xy} - b_{xy} = \langle \Pi f_x, \Pi f_y \rangle$. Consequently

$$|K_\eta(x-y)^2 - \langle Pf_x, Pf_y \rangle^2| \leq 2\sqrt{\ell_x \ell_y} \leq \ell_x + \ell_y.$$

Suppose we select G_0 vertex-disjoint triples of distinct real locations, each satisfying

$$\sum_{\{x,y\} \text{ in the triple}} K_\eta(x-y)^2 \geq d \quad \text{for some fixed } 0 < d < 1.$$

Let $\mathcal{E}$ be the set of the three unordered edges in each selected triple. The compressed real energy expands as

$$\|PF_R P\| * \mathrm{HS}^2 = \sum_x m_x^2 (1-\ell_x)^2 + 2 \sum * \{x,y\} \subset \Omega \cap \mathbb{R} m_x m_y b_{xy}^2,$$

where the second sum runs over all two-element subsets. Retain the selected edges, discard their factors $m_x m_y \geq 1$ from the nonnegative squares, and only then apply the edge-loss estimate. The result is

$$\|PF_R P\| * \mathrm{HS}^2 \geq (2h+1)(N - N_C) - h(h+1)D_R - 2(h+1)\mathcal{L} + 2dG_0 - 2\sum * \{x,y\} \in \mathcal{E}(\ell_x + \ell_y).$$

Each selected vertex has degree two, so $\sum_{\{x,y\}\in\mathcal{E}}(\ell_x + \ell_y) = 2\sum_{\text{selected } x} \ell_x \leq 2\mathcal{L}$. Thus the diagonal loss is $2(h+1)\mathcal{L}$ and the ordered cross-term loss is $4\mathcal{L}$. We conclude that

$$E \geq (2h+1)(N - N_C) - h(h+1)D_R + 2dG_0 - (2h+6)\mathcal{L}. \tag{50}$$

Step 4: packing the height interval. Suppose that the real locations lie in $[A_0, A_0 + H_0]$ and that the kernel K_η satisfies the preceding triple bound with constant d whenever the diameter is at most R. Assign a real location x to cell $\lfloor (x - A_0)/R \rfloor$. There are at most $\lfloor H_0/R \rfloor + 1 \leq H_0/R + 1$ cells, each of diameter at most R; if H_0/R is an integer, this includes the possible final singleton endpoint. Taking $\lfloor s/3 \rfloor$ triples in a cell containing s distinct locations and using $\lfloor s/3 \rfloor \geq s/3 - 2/3$ yields

$$G_0 \geq \frac{D_R}{3} - \frac{2H_0}{3R} - \frac{2}{3} \geq \frac{D}{3} - \frac{N_C}{3} - \frac{2H_0}{3R} - \frac{2}{3}.$$

The nonreal defect (5) gives $\Delta_h := E - (2h+1)N + h(h+1)D \geq N_C$. Before discarding terms, (50) and $D = D_R + 2c_1$ give

$$\Delta_h \geq -(2h+1)N_C + 2h(h+1)c_1 + 2dG_0 - (2h+6)\mathcal{L}.$$

Substitute (49), use the packing bound, and discard the nonnegative $2h(h+1)c_1$ term. This gives

$$\begin{aligned} \Delta_h \geq\ & 2d\left(\frac{D}{3} - \frac{2H_0}{3R} - \frac{2}{3}\right) - (2h+6)\sqrt{N_C E} \\ & + \left(5 - \frac{2d}{3}\right) N_C. \end{aligned} \tag{51}$$

Here $5 = (2h+6) - (2h+1)$, and the subtraction $2d/3$ comes from the nonreal term in the packing bound. The coefficient $5 - 2d/3$ is positive. These inequalities also cover an empty real set, fewer than three real locations, and dependent spanning vectors: use $G_0 = 0$ when no triple exists, and note that dependence only lowers the rank of Π. A negative packing lower bound is harmless.

Step 5: fixed smooth kernels and the limiting bound. For the zero transformation $z_\rho = i(\rho - \frac{1}{2}) \log T/(2\pi)$, the symmetry $\rho \mapsto 1-\bar\rho$ becomes conjugation and preserves the positive-height window and multiplicities. Real elements correspond to $\operatorname{Re}\rho = 1/2$ and lie in $[-T\log T/(2\pi), 0)$. Thus we may take $H_0 = T\log T/(2\pi)$; the fixed-field zero-counting asymptotic gives $H_0/N_K(T) \to 1/n$. We take $h = n$ and $R = 3$, so $\delta_R = \delta$ for the endpoint kernel. First note that the limiting scalar parameters satisfy $q_0 > 0$: the optimized bound is strictly larger than (2), and $r \le n$, so

$$q_0 = \frac{b_0}{3} - \frac{2}{9n} > \frac{3n+2}{9n(n+1)} - \frac{2}{9n} = \frac{1}{9(n+1)} > 0.$$

We apply the analytic formula only to the fixed smooth cutoffs $p_j = \eta_j^2 \to p_\lambda$ in $L^1 \cap L^2$ constructed in Section 5. Each η_j is real, even, smooth, supported in $(-1/2, 1/2)$, and normalized by $\int \eta_j^2 = 1$. Put $K_j = \widehat{p}_j$ and $e_j = \|p_j - p_\lambda\|_1$. Then $C_j = J_{n,r}(p_j) \to C$, $e_j \to 0$, and $\sup_{t\in\mathbb{R}} |K_j(t) - K_\lambda(t)| \le e_j$. For $u, v \ge 0$ with $u + v \le 3$, the reverse triangle inequality in $\mathbb{R}^3$ gives

$$\Big(K_j(u)^2 + K_j(v)^2 + K_j(u+v)^2\Big)^{1/2} \ge \sqrt{\delta} - \sqrt{3}\, e_j.$$

Choose j sufficiently large that $\sqrt{3}\, e_j < \sqrt{\delta}$. Only after this choice may we square to obtain a positive triple bound with

$$\delta_j = \Big(\sqrt{\delta} - \sqrt{3}\, e_j\Big)^2 \longrightarrow \delta.$$

Put

$$b_j = \frac{2n+1-C_j}{n(n+1)}, \quad q_j = \frac{b_j}{3} - \frac{2}{9n}, \quad a_j = 2\delta_j q_j, \quad B_j = (2n+6)\sqrt{C_j}.$$

For all sufficiently large j, we have $q_j > 0$ and $0 < \delta_j < 1$. Fix such a j before letting T tend to infinity. Write E_j for its finite energy, abbreviate $N = N_K(T)$ and $D = D_K(T)$, and put $\Delta_{n,j} = E_j - (2n+1)N + n(n+1)D$. The unweighted energy formula of Section 5 and Lemma 3 give $E_j/N = C_j + o_j(1)$ and $D/N \ge b_j + o_j(1)$. Apply Steps 2–4 to this same smooth kernel, with $d = \delta_j$. With $x = N_C/N \in [0,1]$, drop the positive final term in (51) to get

$$\frac{\Delta_{n,j}}{N} \ge 2\delta_j \left(\frac{D}{3N} - \frac{2H_0}{9N} - \frac{2}{3N}\right) - (2n+6)\sqrt{xE_j/N} \ge a_j - B_j\sqrt{x} + o_j(1).$$

Together with the exact defect bound $\Delta_{n,j}/N \ge x$, this yields

$$\frac{\Delta_{n,j}}{N} \ge \max\{x, a_j - B_j\sqrt{x}\} + o_j(1).$$

The error here is uniform in $x \in [0,1]$; for example,

$$\left|\sqrt{xE_j/N} - \sqrt{xC_j}\right| \le \left|\sqrt{E_j/N} - \sqrt{C_j}\right| = o_j(1).$$

The first branch x is increasing and the second branch $a_j - B_j\sqrt{x}$ is decreasing. Since $a_j, B_j > 0$, they meet once on $x \ge 0$. Writing $y = \sqrt{x}$, the intersection solves $y^2 + B_j y - a_j = 0$ and has $y = 2a_j/(\sqrt{B_j^2 + 4a_j} + B_j)$. Thus

$$\min_{x\ge 0} \max\{x, a_j - B_j\sqrt{x}\} = \left(\frac{2a_j}{\sqrt{B_j^2 + 4a_j} + B_j}\right)^2.$$

Minimizing on $[0, \infty)$ instead of the actual range $[0, 1]$ can only weaken the lower bound. Finally the defining identity for $\Delta_{n,j}$ gives

$$\frac{D}{N} = \frac{2n + 1 - E_j/N}{n(n + 1)} + \frac{\Delta_{n,j}/N}{n(n + 1)} = b_j + \frac{\Delta_{n,j}/N}{n(n + 1)} + o_j(1).$$

Consequently, for every such fixed j,

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq b_j + \frac{1}{n(n + 1)} \left(\frac{2a_j}{\sqrt{B_j^2 + 4a_j} + B_j} \right)^2.$$

Now let $j \to \infty$ on the right. Continuity at (C, b_0, q_0, δ) proves (46). This order of limits asserts no asymptotic for the energy of the nonsmooth endpoint profile and requires no endpoint-support pair theorem. >

Step 6: an elementary certificate for the quadratic increment. Every quadratic field is Galois, so $n = r = 2$ and $\lambda = 1/2$. Section 5 gives $C < 7/3$, $b_0 > 4/9$, and hence $q_0 > 1/27$. The elementary bounds $3 < \pi < 22/7$, $\sin(1/2) < 1/2$, and $\cos(1/2) > 1 - (1/2)^2/2 = 7/8$ give $c < 2/21 < 1/10$. The first alternative defining δ is larger than $(49/1887)^2 > 1/25000$, while the second is larger than

$$\frac{3/8}{81(22/7)^4(101/100)^3} = \frac{37515625}{814570394814} > \frac{1}{25000}.$$

Hence $\delta > 1/25000$. For the limiting numerical minimum in (46), if $x \geq 10^{-14}$ then $\max\{x, a - B\sqrt{x}\} \geq 10^{-14}$. If $0 \leq x < 10^{-14}$, then $C < 7/3 < 4$ implies $10\sqrt{Cx} < 2 \cdot 10^{-6}$, and therefore

$$2\delta q_0 - 10\sqrt{Cx} > \frac{2}{675000} - 2 \cdot 10^{-6} > 9 \cdot 10^{-7} > 10^{-14}.$$

Thus $\min_{x\geq 0} \max\{x, a - B\sqrt{x}\} \geq 10^{-14}$. Equation (46) gives an increment $10^{-14}/6 > 10^{-15}$. Since $C = \frac{1}{2} + \cot(1/2)$, we have $b_0 = (9/2 - \cot(1/2))/6$, proving (47). $\square$

This explicit improvement is small. Its role is to show that retaining height-window geometry can go strictly beyond the optimized common-kernel moment certificate, while controlling off-line zeros without GRH. It is a fixed-field asymptotic statement: it supplies neither an explicit height from which the improvement is visible nor a uniform error as the field varies.

8 Comparisons and limitations

We separate three kinds of assertion: bounds explicitly stated in earlier sources, elementary consequences of those bounds, and the unconditional results proved in this paper. Throughout, the field is fixed as $T \to \infty$, and all Dedekind-zero proportions have the multiplicity-weighted count $N_K(T)$ as denominator. Conditional comparisons below assume GRH; they are not unconditional benchmarks.

8.1 Previously available unconditional quadratic benchmarks

For a fixed quadratic field K, the factorization

$$\zeta_K(s) = \zeta(s)L(s, \chi_K)$$

has a nonprincipal primitive quadratic character χ_K. Since $L(s, \chi_K)$ is entire, every nontrivial zero of ζ remains a zero of ζ_K, even if the two factors vanish at the same point. Writing $D_\zeta(T)$ for the number of distinct Riemann-zeta zero locations and $N_\zeta(T)$ for its zero count with multiplicity, we have

$$D_K(T) \geq D_\zeta(T), \qquad N_K(T) \sim 2N_\zeta(T).$$

Consequently any unconditional bound $\liminf D_\zeta(T)/N_\zeta(T) \geq \alpha$ implies

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq \frac{\alpha}{2}. \tag{52}$$

Indeed, $D_K/N_K \geq (D_\zeta/N_\zeta)(N_\zeta/N_K)$ and the second factor tends to $1/2$, while $0 \leq D_\zeta/N_\zeta \leq 1$. This counts distinct complex locations, not simple zeros of the product.

Two historical dates must be distinguished. As recorded in Lamzouri's introduction [1], Ki and Lee's 2012 Theorem 5 [6] gave $\liminf D_\zeta/N_\zeta > 0.70$ without RH. Thus it implies an unconditional quadratic Dedekind proportion *greater than* 35%. This is an elementary consequence of their Riemann-zeta theorem, not a quadratic Dedekind theorem explicitly stated in that paper.

The 2026 result of Alpöge–Furman [14, Theorem A], reproved by Lamzouri [1, Theorem 1.1], gives

$$\liminf \frac{D_\zeta(T)}{N_\zeta(T)} \geq C_1, \qquad C_1 = \frac{1+c_0}{2} = 0.83625035183970582\ldots,$$

with c_0 as in Corollary 2. Therefore, before the present Dedekind extension, (52) already supplies

$$\boxed{\liminf \frac{D_K(T)}{N_K(T)} \geq \frac{C_1}{2} = \frac{1+c_0}{4} = 0.41812517591985291\ldots.}$$

This is approximately 41.81251759%. It uses the 2026 result and must not be described as information supplied by the 2019 paper.

The present quadratic scalar bound is $44.49187130479246\ldots\%$, an increase of $2.67935371280717\ldots$ *percentage points* over this inherited 2026 benchmark. The geometric increment in (47) is additional and is not included in these scalar figures.

Scope of the comparison. These are consequences of the cited results, not a classification of all previously available unconditional Dedekind bounds. In particular, the comparison alone does not establish a literature-wide record.

8.2 Alsharif's thesis and the seven-author project

We use the full institutional copies of [19, 20]. Alsharif's title page is dated December 2018 and the repository records the degree in 2019. Tripp's dissertation is dated 2022; its Chapter VI attributes the joint theorems to the seven authors with the date 2020. The latter is an attribution date, not evidence of a public 2020 manuscript. Chapter VI is a four-page summary of results, not a full proof of the seven-author project. We compare its public statements and explicitly identify consequences that use them; this is not an independent verification of proofs not included there. For the following comparisons, $N_K(T)$ counts zeros with multiplicity, $D_K(T)$ counts distinct locations, and, writing μ_ρ for a location's multiplicity,

$$N_K^*(T) = \sum_{\substack{\rho \text{ distinct} \\ 0<\Im\rho\leq T}} \mu_\rho^2.$$

For the same height window, this agrees with the sources' sum of μ_ρ over zeros listed with multiplicity. Under GRH there is no distinction between distinct ordinates and distinct complex locations. The endpoint convention is reconciled below.

The Galois precedent and its elementary consequences. Alsharif's Theorem 1.7 and Corollary 1.8 state, for a Galois field of degree n under GRH, a strict-support pair formula and

$$N_K^*(T) \le \left(n + \frac{1}{3} + o_K(1)\right) N_K(T), \qquad D_K(T) \ge \left(\frac{3}{3n+1} + o_K(1)\right) N_K(T).$$

Evaluating the stated distinct-zero formula gives $42.8571428\ldots\%$ in degree two, 30% in degree three, and $23.0769230\ldots\%$ in degree four. The proof uses a squared-sinc test and Cauchy–Schwarz [19, Chapter 3]. Its analytic ingredients include [22, equation (30)] and the Galois prime-coefficient calculation [23, Lemma 5.2], as identified in [19, Proposition 4.1 and Lemma 5.1].

An elementary improvement already follows from this GRH moment bound. For integers $\mu, k \ge 1$, either $\mu \le k$ or $\mu \ge k+1$, so $(\mu - k)(\mu - k - 1) \ge 0$. Summing over distinct locations gives

$$D_K(T) \ge \frac{(2k+1)N_K(T) - N_K^*(T)}{k(k+1)} \quad (k \in \mathbb{Z}_{\ge 1}). \tag{53}$$

This finite inequality itself needs no GRH. The hypothesis enters only through an upper bound for $N_K^*(T)$. At $k = n$, Alsharif's moment estimate thus implies under GRH

$$\liminf \frac{D_K(T)}{N_K(T)} \ge B_n := \frac{3n+2}{3n(n+1)}, \qquad B_n - \frac{3}{3n+1} = \frac{2}{3n(n+1)(3n+1)} > 0.$$

Consequently B_n is not a new GRH consequence. The present scalar bound $b_{n,n}$ is stronger than B_n and is unconditional, but its optimized constant is inherited from the variational theory cited above.

Normalization and analytic scope. Both theses use the natural degree normalization. In the positive-height convention of this paper, write

$$F_K^{\mathrm{nat}}(\alpha, T) = \left(\frac{nT \log T}{2\pi}\right)^{-1} \sum_{0 < \gamma, \gamma' \le T} T^{in\alpha(\gamma - \gamma')} w(\gamma - \gamma'), \qquad w(v) = \frac{4}{4 + v^2},$$

with multiplicities. Their support $|\alpha| < 1/n$ corresponds to our strict support $|u| < 1$ through $u = n\alpha$. Alsharif's Definition 1.5 includes the lower endpoint $\gamma = 0$. Under GRH this adds only the finitely many copies of a possible central zero. The resulting change in F_K^{nat} is $O_K((T \log T)^{-1})$, uniformly for real α: its unnormalized absolute value is bounded by a constant times $1 + \sum_{\gamma > 0} w(\gamma) = O_K(1)$, using the local zero bound. It changes each of N_K, D_K, and N_K^* by $O_K(1)$ as well. Thus the endpoint convention does not affect the comparisons.

For a real even test $f \in C_c^\infty((-1,1))$, under GRH the Fourier convention of Section 4 gives the exact finite-sum identity

$$\left(\frac{nT \log T}{2\pi}\right)^{-1} \sum_{0 < \gamma, \gamma' \le T} \widehat{f}\left(\frac{(\gamma - \gamma') \log T}{2\pi}\right) w(\gamma - \gamma') = \int_{\mathbb{R}} f(u) F_K^{\mathrm{nat}}(u/n, T)\, du.$$

Indeed, interchanging the finite sum and integral gives $\widehat{f}(-(\gamma - \gamma') \log T/(2\pi))$; evenness of f removes the minus sign. Alsharif's Galois pair formula has main terms $nT^{-2n|\alpha|} \log T + n|\alpha|$. After this change of scale their integrated contribution tends to $nf(0) + \int_{\mathbb{R}} |u| f(u)\, du$. To check the concentrated first term, put $L = \log T$ and change variables $v = 2Lu$:

$$nL \int_{\mathbb{R}} f(u) e^{-2L|u|}\, du = \frac{n}{2} \int_{\mathbb{R}} f\left(\frac{v}{2L}\right) e^{-|v|}\, dv \longrightarrow nf(0)$$

by dominated convergence. The second term is exactly $\int_{\mathbb{R}} |u| f(u)\, du$. These are the main terms in (7) when $r = n$; passing an error term through the integral requires uniformity or an integrable error bound, not merely pointwise convergence. The moment comparison above uses Alsharif's stated Corollary 1.8 directly, so it does not depend on inferring such an error bound from his displayed pair formula. There is no support enlargement merely from rescaling. Moreover, [18, Section 1] already uses the $\log T$ scale of this paper.

Theorem VI.2.1 of [20] states, under GRH, the abelian asymptotic

$$F_K^{\mathrm{nat}}(\alpha, T) = (n + o_K(1))T^{-2n|\alpha|} \log T + n|\alpha| + o_K(1) \quad (|\alpha| \le 1/n),$$

uniformly on the indicated interval, and states the corresponding upper bound for arbitrary extensions. The uniform abelian asymptotic can be integrated against the fixed test f: the error in the concentrated term is $o_K(1)$ after integration, since $L \int_{\mathbb{R}} |f(u)| e^{-2L|u|}\, du \le \|f\|_\infty$, and the remaining uniform error contributes $o_K(1)\|f\|_1$. For an upper bound, integration in the same direction would require $f \ge 0$, as well as appropriate control of its error. We do not infer an equality or a signed-test estimate from the arbitrary-field upper bound; the arbitrary-field numerical comparisons below instead use the moment theorem stated separately in that chapter. In contrast, (7) is an equality with coefficient r/n in front of $\int |u| f(u)\, du$, for every fixed field, and retains the real parts of the zeros without GRH. Since $r \le n$, it also distinguishes fields that the degree-only upper bound does not. These are differences of statements, not claims that the classical coefficient calculations or the GRH support range originate here. The same chapter also reports a special dihedral asymptotic in an unnumbered remark on p. 116. Thus case-specific nonabelian asymptotics are also part of the prior project; we do not claim otherwise. That remark's coefficient and group-order convention are not used in the numerical comparisons here, which use the numbered statements.

Previously stated arbitrary-field proportions. Theorem VI.2.2 of [20] states under GRH, for every degree-n field, $N_K^*(T) \le (c_n + o_K(1))N_K(T)$ with

$$c_2 = 2.3226, \qquad c_3 = 3.3232, \qquad c_4 = 4.3235.$$

It also states the general choices $n + 1/3$ and $(1 + 10^{-10})n + 0.3243$. Corollary VI.2.3 states the reciprocal bound $D_K/N_K \ge 1/c_n + o_K(1)$ and the additional quadratic bound 45.85%. In degrees three and four the reciprocal bounds are $30.0914780\ldots\%$ and $23.1294090\ldots\%$. Thus arbitrary-field positive proportions under GRH are explicit precedents, not missing cases supplied for the first time by this paper. The following comparison also retains the stronger elementary consequences of their moment estimates.

8.3 Comparison with the GRH moment estimates

For abelian extensions, the explicit distinct-zero bounds in [18, Corollary 1.3] are 45.85%, 27.94%, and 11.27% in degrees 2, 3, and 4, respectively, all assuming GRH. The 2019 abelian-field SDP estimates [18], under GRH, bound the squared-multiplicity mean by $2.3226, 3.3232, 4.3235$ in degrees $2, 3, 4$. The same estimates are stated for arbitrary fields in [20, Theorem VI.2.2]. Equation (53), with $k = n$, gives the integer-moment GRH consequences below, also for arbitrary fields if that stated theorem is used. These are elementary consequences, not the distinct-zero corollaries printed in either source. The entries are proportions, not percentages. The present column gives the degree-only scalar bound $b_{n,n}$, without the rank-sensitive or geometric refinements; the GRH column is $(2n + 1 - c_n)/(n(n + 1))$, using the reported moment constants c_n.

Degree	Unconditional scalar bound	GRH moment consequence
2	$0.4449187130\ldots$	$0.4462333333\ldots$
3	$0.3057123676\ldots$	0.3064000000
4	$0.2334036149\ldots$	0.2338250000

For the quadratic comparison there is a stronger consequence, already used in the proof of [18, Corollary 1.3, Section 3.2]. It is important here to use the diagonal multiplicity moment N_K^*, not to substitute it for the kernel energy E in (4). For every positive integer μ,

$$(\mu-2)(\mu-3) \geq 2\mathbf{1}_{\{\mu=1\}}:$$

at $\mu = 1$ there is equality, at $\mu = 2, 3$ both sides vanish, and at $\mu \geq 4$ the left side is nonnegative. Summing over distinct locations gives the exact inequality

$$N_K^*(T) - 5N_K(T) + 6D_K(T) \geq 2N_K^{\text{simple}}(T).$$

For a fixed quadratic field, the GRH estimate $N_K^{\text{simple}}(T)/N_K(T) \geq 1/27 + o_K(1)$ is given in [16, Theorem 1 and the following Remark, pp. 87–89]. Here we use the remark under RH for ζ_K, not the theorem's $1/54$ bound under RH for ζ alone. Combining the $1/27$ bound with the preceding finite inequality and $c_2 = 2.3226$ yields, under GRH,

$$\liminf_{T\to\infty} \frac{D_K(T)}{N_K(T)} \geq \frac{5-2.3226}{6} + \frac{1}{3\cdot 27} = 0.4585790123\ldots.$$

This is the same argument used in [18, Section 3.2], evaluated with its reported moment constant to more decimal places; it is not a new conditional improvement over that proof. Our result removes GRH; it is not numerically better than all bounds under that additional assumption. Relative to a direct use of Lamzouri's original $k = 1$ certificate, the quadratic bound increases from $0.3347561391\ldots$ to $0.4449187130\ldots$. For $n \geq 3$, the $k = 1$ lower bound $(3 - C_{n,r})/2$ is negative, since $C_{n,r} > n$, whereas the $k = n$ bound is positive because $C_{n,r} < n + 1/3 < 2n + 1$.

8.4 A sharp finite model and limitations of second-moment data

The following exact construction shows why retaining factor labels does not automatically strengthen the scalar certificate at the flat-kernel cost. Fix an integer $n \geq 2$. Regard the n coordinates of a multiplicity vector as n colors, let Z_i be the finite real multiset in color i, and put

$$N_i = |Z_i|, \qquad E_{ij} = \sum_{z\in Z_i}\sum_{w\in Z_j} K(z-w)^2.$$

The union here is a multiset sum: multiplicities at coincident locations add. Its scalar energy is therefore $E = \sum_{i,j} E_{ij}$. Take the flat profile $\eta_{\text{flat}} = \mathbf{1}_{[-1/2,1/2]}$, whose squared L^2 norm is one. Its kernel is $K(z) = \widehat{\eta_{\text{flat}}^2}(z) = \sin(\pi z)/(\pi z)$, with the removable value $K(0) = 1$. This profile is admissible for the finite lemma in Section 3; it is not a smooth strict-support test for the analytic pair formula. Place all locations at mutually distinct integers and use the following multiplicity vectors, where e_i is the ith standard basis vector:

Multiplicity vector	Number of locations
$(1,\ldots,1)$	$n(2n+1)$
$(1,\ldots,1)+e_i$	n for each i
ne_i	1 for each i

For a fixed color i, direct summation over the three rows gives

$$\begin{aligned} N_i &= n(2n+1) + \big(2n + n(n-1)\big) + n = 3n(n+1), \\ E_{ii} &= n(2n+1) + \big(4n + n(n-1)\big) + n^2 = 4n(n+1), \\ E_{ij} &= n(2n+1) + \big(2n + 2n + n(n-2)\big) = 3n(n+1) \quad (i \neq j). \end{aligned}$$

Here different integer locations are orthogonal because $K(m) = 0$ for every nonzero integer m. Thus $E_{ii} = \frac{4}{3}N_i$ and $E_{ij} = N_i$ for $i \neq j$. Equivalently, $(E_{ij}) = n(n+1)(I_n + 3J_n)$, where I_n is the identity matrix and every entry of J_n is one. The union has

$$N = 3n^2(n+1), \qquad D = n(3n+2), \qquad \frac{E}{N} = n + \frac{1}{3},$$

and consequently

$$D = \frac{(2n+1)N - E}{n(n+1)}.$$

Indeed, every occupied location has total multiplicity n or $n+1$, so $(\mu - n)(\mu - n - 1) = 0$ there, and orthogonality gives $E = \sum \mu^2$. This proves exact equality in the $k = n$ certificate. Consequently the specified masses and full colored second-moment matrix alone cannot force a larger distinct-location count for every finite colored real multiset. Repeating the multiplicity pattern at unused integers produces arbitrarily large models with the same ratios.

This is a finite abstract model, not a construction of Dedekind zeros. It does not realize all the analytic constraints on those zeros and does not establish an obstruction at the optimized cosine-kernel cost. In particular, it does not contradict Theorem 13: the optimized cosine kernel has a positive three-point gap, whereas the flat kernel vanishes at all nonzero integers. Stronger multiple-kernel methods are not excluded either.

Finally, as explained in Section 6, the simple-critical certificate $S \geq 2N - E$ with the present scalar energy bound yields only $2 - C_{n,r} < 0$ for $n \geq 2$. It therefore supplies no positive simple-critical proportion for the Dedekind product. Transferring simplicity from holomorphic factors also requires control of common zeros, since their local multiplicities add. These are limitations of the stated arguments, not an impossibility theorem for other methods.

Use of AI: The author prepared a 6-page draft (without using AI) with outlines of proofs extending Lamzouri's method to Dedekind zeta functions. The author then fed this draft to multiple AI tools (Claude, ChatGPT and two open-weight models with open-source harnesses) to work out the details of the equations and identify missing gaps. The author fixed those missing gaps and iteratively verified and expanded the proofs using AI tools. The author is solely responsible for the correctness of the proofs.

References

[1] Y. Lamzouri, *A new proof that more than* 2/3 *of the zeros of the Riemann zeta function are simple and on the critical line*, 2026. arXiv:2609.02882.

[2] N. Levinson, *More than one third of zeros of Riemann's zeta-function are on* $\sigma = 1/2$, Adv. Math. **13** (1974), 383–436. doi:10.1016/0001-8708(74)90074-7.

[3] J. B. Conrey, *More than two fifths of the zeros of the Riemann zeta function are on the critical line*, J. Reine Angew. Math. **399** (1989), 1–26. doi:10.1515/crll.1989.399.1.

[4] K. Pratt, N. Robles, A. Zaharescu and D. Zeindler, *More than five-twelfths of the zeros of* ζ *are on the critical line*, Res. Math. Sci. **7** (2020), Article 2. doi:10.1007/s40687-019-0199-8.

[5] D. W. Farmer, *Counting distinct zeros of the Riemann zeta-function*, Electron. J. Combin. **2** (1995), R1. doi:10.37236/1195.

[6] H. Ki and Y. Lee, *Zeros of the derivatives of the Riemann zeta-function*, Funct. Approx. Comment. Math. **47** (2012), no. 1, 79–87. doi:10.7169/facm/2012.47.1.6.

[7] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, in *Analytic Number Theory* (St. Louis, 1972), Proc. Sympos. Pure Math. **24**, Amer. Math. Soc., Providence, RI, 1973, 181–193. Author's PDF.

[8] H. L. Montgomery, *Distribution of the zeros of the Riemann zeta function*, in *Proceedings of the International Congress of Mathematicians* (Vancouver, 1974), Vol. 1, Canadian Math. Congress, Montreal, 1975, 379–381. Reports joint optimization with M. E. Taylor. Author's publication list.

[9] A. Chirre, F. Gonçalves and D. de Laat, *Pair correlation estimates for the zeros of the zeta function via semidefinite programming*, Adv. Math. **361** (2020), 106926. doi:10.1016/j.aim.2019.106926.

[10] F. Aryan, *On an extension of the Landau–Gonek formula*, J. Number Theory **233** (2022), 389–404. doi:10.1016/j.jnt.2021.06.015.

[11] S. A. C. Baluyot, D. A. Goldston, A. I. Suriajaya and C. L. Turnage-Butterbaugh, *An unconditional Montgomery theorem for pair correlation of zeros of the Riemann zeta-function*, Acta Arith. **214** (2024), 357–376. doi:10.4064/aa230612-20-3.

[12] S. A. C. Baluyot, D. A. Goldston, A. I. Suriajaya and C. L. Turnage-Butterbaugh, *Pair Correlation of Zeros of the Riemann Zeta Function I: Proportions of Simple Zeros and Critical Zeros*, 2025. arXiv:2501.14545v2; Sections 2–3 correct the pointwise error in [11].

[13] E. Hasanalizade, Q. Shen and P.-J. Wong, *Counting zeros of Dedekind zeta functions*, 2021. arXiv:2102.04663, Section 2.1.

[14] L. Alpöge and R. Furman, *More than two thirds of the zeta zeros are simple and on the critical line*, 2026. arXiv:2608.13637.

[15] J. B. Conrey, A. Ghosh and S. M. Gonek, *Simple zeros of the zeta function of a quadratic number field. I*, Invent. Math. **86** (1986), 563–576. doi:10.1007/BF01389269.

[16] J. B. Conrey, A. Ghosh and S. M. Gonek, *Simple zeros of the zeta-function of a quadratic number field, II*, in A. C. Adolphson, J. B. Conrey, A. Ghosh and R. I. Yager (eds.), *Analytic Number Theory and Diophantine Problems* (Stillwater, OK, 1984), Progr. Math. **70**, Birkhäuser, Boston, 1987, 87–114. doi:10.1007/978-1-4612-4816-3_5.

[17] X. Wu and L. Zhao, *On simple zeros of the Dedekind zeta-function of a quadratic number field*, Mathematika **65** (2019), no. 4, 851–861. doi:10.1112/S0025579319000196.

[18] D. de Laat, L. Rolen, Z. Tripp and I. Wagner, *Pair correlation for Dedekind zeta functions of abelian extensions*, 2019. arXiv:1908.04876.

[19] M. Alsharif, *Zeros of the Dedekind Zeta-Function*, M.S. thesis, University of Mississippi, 2019, Electronic Theses and Dissertations, 1542. Title page dated December 2018. Institutional record and full thesis; Theorem 1.7 and Corollary 1.8, pp. 9–10; Chapters 3–5.

[20] Z. D. Tripp, *Ranks, Cranks, and Automorphic Forms*, Ph.D. dissertation, Vanderbilt University, 2022. Institutional record and full dissertation; Chapter VI, pp. 114–117, especially Theorems VI.2.1–VI.2.2 and Corollary VI.2.3. The chapter attributes these joint results to Alsharif–Gibson–de Laat–Milinovich–Rolen–Tripp–Wagner, 2020.

[21] M. Alsharif, M. Gibson, D. de Laat, M. Milinovich, L. Rolen, Z. Tripp and I. Wagner, *Pair correlation for Dedekind zeta functions*, listed as in preparation on L. Rolen's publication list, accessed September 7, 2026. This list is cited for the project's status; public statements of the joint results are cited through [20, Chapter VI].

[22] M. R. Murty and A. Perelli, *The pair correlation of zeros of functions in the Selberg class*, Int. Math. Res. Not. **1999** (1999), no. 10, 531–545. doi:10.1155/S1073792899000276.

[23] M. B. Milinovich and C. L. Turnage-Butterbaugh, *Moments of products of automorphic L-functions*, J. Number Theory **139** (2014), 175–204. doi:10.1016/j.jnt.2013.12.012.

[24] R. Garunkštis and J. Paliulionytė, *Pair correlation of zeros of Dirichlet L-functions and proportion of simple zeros*, Glasnik Mat. **60** (2025), no. 2, 207–227. doi:10.3336/gm.60.2.03.

[25] V. Chandee, Y. Lee, S.-C. Liu and M. Radziwiłł, *Simple zeros of primitive Dirichlet L-functions and the asymptotic large sieve*, Quart. J. Math. **65** (2014), no. 1, 63–87. doi:10.1093/qmath/hat008.

[26] D. Hu, I. Kaneko, S. Martin and C. Schildkraut, *Order of zeros of Dedekind zeta functions*, Proc. Amer. Math. Soc. **150** (2022), no. 12, 5111–5120; see also Lemma 3.1. doi:10.1090/proc/16041.

[27] H. M. Stark, *Some effective cases of the Brauer–Siegel theorem*, Invent. Math. **23** (1974), 135–152. doi:10.1007/BF01405166.

[28] M. K. Das, T. Ismoilov and A. P. Ramos, *Fourier optimization and pair correlation problems*, 2025. arXiv:2502.05106.

[29] B. Winckler, *Théorème de Chebotarev effectif*, 2013. arXiv:1311.5715, Theorem 1.1.

[30] J. Thorner and A. Zaman, *A unified and improved Chebotarev density theorem*, Algebra Number Theory **13** (2019), no. 5, 1039–1068. doi:10.2140/ant.2019.13.1039.

[31] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) **8** (1974), 73–82. doi:10.1112/jlms/s2-8.1.73.