

A cubic lower bound for electrostatic equilibria: Failure of the real-charge Morse Maxwell bound

Shiva Kintali*

August 10, 2026

Abstract

For distinct sites $a_1, \dots, a_n \in \mathbb{R}^3$ and nonzero real charges $q_1, \dots, q_n$, let $V_{a,q}(x) = \sum_{i=1}^n \frac{q_i}{|x - a_i|}$, and let $M(n)$ be the supremum of $\#\text{Crit}(V_{a,q})$ over configurations with finite critical set. We prove

$$M(n) \geq \left(\frac{1}{48} - o(1) \right) n^3.$$

Every final configuration supplied by our construction contains at least one positive and at least one negative charge. In particular, $M(n) > (n-1)^2$ for $n = 41$ and for every $n \geq 43$. All equilibria in the final configurations are nondegenerate, so these configurations form an infinite family of violations of the real-charge Morse Maxwell bound.

Our construction starts with charges on two perpendicular rays. The ray carrying the even-degree terms uses $2\lfloor L/2 \rfloor$ charges, and the ray carrying the odd-degree terms uses $2\lceil L/2 \rceil - 1$ charges; the total is $2L - 1$. A charge on a ray contributes, in every degree, only the solid zonal harmonic about that ray. Alternating the two axes between consecutive degrees and separating the degrees onto different spatial scales makes the local model at scale k the sum of perpendicular solid zonal harmonics of degrees k and $k+1$. Its off-plane critical points arise from pairs (α, β) of zeros of P'_k and P'_{k+1} satisfying $\alpha^2 + \beta^2 < 1$ and a radial sign condition. The sign condition selects one member of each antipodal pair, while reflection across the coordinate plane produces two points. Consequently their total number is $g_k = (\frac{1}{2} + o(1))k^2$, and all these points are nondegenerate. Each local model also has a nondegenerate critical point on the axis of its odd-degree summand. A small generic perturbation then gives nonzero charges of both signs, a nonzero total charge, and a finite Morse critical set without losing any of the counted equilibria.

1 Introduction

1.1 The problem and our main result

Let $a_1, \dots, a_n \in \mathbb{R}^3$ be pairwise distinct sites and let $q_1, \dots, q_n \in \mathbb{R} \setminus \{0\}$ be their charge strengths. On the charge-free domain $\Omega_a = \mathbb{R}^3 \setminus \{a_1, \dots, a_n\}$, the Coulomb potential and its equilibrium set are

$$V_{a,q}(x) = \sum_{i=1}^n \frac{q_i}{|x - a_i|}, \quad \text{Crit}(V_{a,q}) = \{x \in \Omega_a : \nabla V_{a,q}(x) = 0\}. \quad (1)$$

*Email: shiva.kintali@gmail.com Homepage: shivakintali.github.io.

Here $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and inner product, respectively. The electric field is $-\nabla V_{a,q}$, so these critical points are precisely the positions at which a nonzero test charge experiences zero force. A critical point is *nondegenerate* if its Hessian is invertible, and its Morse index is the number of negative Hessian eigenvalues, counted with multiplicity. We call the potential *Morse* if all its critical points are nondegenerate.

For each integer $n \geq 1$, define

$$M(n) = \sup \left\{ \# \text{Crit}(V_{a,q}) : \begin{array}{l} a \in (\mathbb{R}^3)^n, \quad a_i \neq a_j \quad (i \neq j), \\ \# \text{Crit}(V_{a,q}) < \infty \end{array}, \quad q \in (\mathbb{R} \setminus \{0\})^n, \right\}. \quad (2)$$

The supremum is taken in the extended nonnegative integers. The finiteness condition excludes configurations with nonisolated equilibrium sets but allows isolated degenerate equilibria. Every final configuration asserted in Theorem 1.1, however, is Morse. Because $V_{a,q}$ is harmonic on Ω_a , the Hessian at every critical point has trace zero. Consequently, every nondegenerate equilibrium in dimension three is a saddle of Morse index 1 or 2.

Maxwell's original discussion used degree-weighted counts of equilibrium points and lines. The upper bound $(n-1)^2$ is the modern isolated-equilibrium bound extracted from that discussion [16, 7]. In this paper, the *real-charge Morse Maxwell bound* means the assertion that every configuration of n distinct sites with nonzero real strengths and a finite Morse critical set satisfies

$$\# \text{Crit}(V_{a,q}) \leq (n-1)^2. \quad (3)$$

This formulation allows strengths of either sign. Its restriction to strictly positive strengths is a separate assertion. We construct configurations with both signs that violate (3) and prove a cubic lower bound for $M(n)$. Their equilibrium counts therefore exceed the quadratic benchmark by an unbounded factor.

To state the quantitative result, for each integer $l \geq 0$ let P_l be the Legendre polynomial normalized by $P_l(1) = 1$. For a nonzero real polynomial f , let $Z(f)$ denote its set of distinct real zeros in $(-1, 1)$. For a unit vector w and $x \neq 0$, set

$$Z_l^w(x) = |x|^l P_l \left(\frac{\langle x, w \rangle}{|x|} \right). \quad (4)$$

The right-hand side extends across the origin to a harmonic homogeneous polynomial of degree l , called the solid zonal harmonic of degree l about the axis w . We write $u = e_3$ and $v = e_1$ for two perpendicular standard coordinate unit vectors.

Theorem 1.1 (Main theorem). *Let $L \geq 2$ and $n \geq 2L - 1$ be integers. There are n pairwise distinct sites in $\mathbb{R}^3$ and nonzero real charges, including at least one positive and at least one negative charge, for which the total charge is nonzero, the critical set is finite, every equilibrium is nondegenerate, and*

$$\# \text{Crit}(V_{a,q}) \geq B_L := S_L + L - 1, \quad S_L := \sum_{k=1}^{L-1} g_k,$$

where

$$g_k := \# \{ (\alpha, \beta) \in Z(P'_k) \times Z(P'_{k+1}) : \alpha^2 + \beta^2 < 1 \}.$$

Since $P'_1 = 1$, one has $g_1 = 0$. Moreover,

$$g_k \geq \frac{k^2}{2} - 5k - 6 \quad (k \geq 3), \quad g_k = \frac{1}{2}k^2 + O(k).$$

In particular, for every $n \geq 3$, $M(n) \geq B_{\lceil n/2 \rceil}$. Consequently, as $n \rightarrow \infty$ through the positive integers,

$$M(n) \geq \left(\frac{1}{48} - o(1) \right) n^3.$$

In particular,

$$M(41) > 40^2, \quad M(n) > (n-1)^2 \quad \text{for every } n \geq 43.$$

Our proof is completed in Section 5. We first construct B_L nondegenerate equilibria and then arrange that the entire critical set is finite and Morse. Thus the theorem disproves (3) itself, even though the definition of $M(n)$ also allows finite degenerate critical sets. Both signs are forced by the vanishing moments used in the construction and are preserved by the final perturbation.

The following values illustrate the finite estimate. All entries in the third column are guaranteed lower bounds; additional equilibria may occur.

n	$L = \lceil n/2 \rceil$	B_L	$(n-1)^2$
41	21	1620	1600
42	21	1620	1681
43	22	1861	1764
44	22	1861	1849
47	24	2411	2116

The finite base cases and induction in Section 5 establish the strict inequalities for $n = 41$ and every $n \geq 43$. At $n = 42$, the displayed bound is below 41^2 and gives no counterexample. This is a limitation of the estimate, not an upper bound on $M(42)$ or a claim that the actual constructed configuration has exactly 1620 equilibria.

For comparison, let $M_+(n)$ denote the quantity in (2) with $(\mathbb{R} \setminus \{0\})^n$ replaced by $(0, \infty)^n$; then $M_+(n) \leq M(n)$. Our positive cubature construction in [14] gives

$$M_+(n) \geq (3.596 \times 10^{-4} - o(1)) n^{3/2}.$$

The cubic lower bound proved here concerns $M(n)$; our construction does not establish a cubic lower bound for $M_+(n)$.

1.2 Motivation

The equilibrium-counting problem lies at the intersection of potential theory, Morse theory, real algebraic geometry, and the geometry of harmonic functions. Even its qualitative aspects require care. The field equations contain algebraic radicals; signed configurations can have positive-dimensional critical sets, even with only three charges [13]; and critical points produced by different asymptotic models must remain separated when those models are combined into one Coulomb potential.

For positive Morse configurations, classical Morse-theoretic arguments give at least $n-1$ equilibria; see, for example [17, 5]. Positivity also confines every equilibrium to the convex hull of the sites: at an equilibrium, the field equation expresses x as the convex combination of the a_i with weights proportional to $q_i/|x-a_i|^3$. Allowing signed charges removes this general confinement principle and permits cancellations between multipole moments. The construction below exploits those cancellations to realize a long alternating zonal jet in a rescaled limit and to

make each pair of consecutive degrees contribute quadratically many critical points. Separating these contributions over $L - 1$ spatial scales is what raises the total count from quadratic to cubic.

1.3 Historical and related work

Maxwell discussed electrostatic equilibrium counting in Article 113 of his treatise [16]. Gabrielov, Novikov, and Shapiro explained how the modern nondegenerate-equilibrium bound $(n - 1)^2$ is extracted from that discussion [7]. Early Morse-theoretic work on nondegenerate Newtonian potentials includes Kiang [11]. Morse and Cairns later proved that, with the strengths and all but one site fixed, nondegeneracy holds for almost every position of the remaining site [17, Theorem 6.2]. Abanov, Hayford, Khavinson, and Teodorescu give a broader account of equilibrium questions in classical electrostatics [1].

The first general finite upper bound for nondegenerate configurations was obtained by Gabrielov, Novikov, and Shapiro using Khovanskii’s fewnomial theory; their bound is independent of the ambient dimension, and their specialized bound for three charges is 12 [7]. Zolotov subsequently used the Thom–Milnor theorem to obtain the exponential bound $5 \cdot 9^{n+3}$ [25]. Edelsbrunner, Fillmore, and Oliveira proved the Bézout bound $2^n(3n - 2)^3$ when the strengths and all but one site are fixed and the remaining site is chosen generically; they also disproved a Voronoi-theoretic upper bound proposed in [7]; see [5]. For three positive charges, Gabrielov, Dm. Novikov, T. Novikov, and Shapiro improved the upper bound for nondegenerate equilibria from 12 to 6 in 2026 [8]; a sharp bound of 4 for arbitrary nonzero real strengths, assuming only that the critical set is finite, is proved in [13].

Several sharper or more explicit results are known under planar, symmetric, or few-charge hypotheses. Killian obtained an improved upper bound for equilibria in the plane of coplanar charges [12], and Peretz treated the three-charge problem by the argument principle [18]. Bilogliadov analyzed equal positive charges at roots of unity for a family of Riesz potentials [4]. Tsai proved the proposed bound for equilateral and isosceles-right three-site geometries [22] and later treated three charges of equal absolute magnitude [23]. Uteshev and Yashina developed exact algebraic and bifurcation methods for three and four positive planar charges [24], while Erdélyi–Rosenblatt–Rosenblatt studied the structure and finiteness of field zero sets, including collinear planar sources [6]. Lee and Tsai exhibited four coplanar positive charges with nine planar equilibria, attaining $(4 - 1)^2$ in that restricted setting [15]. Ghosh, Goldberg, and Hollender studied the computational complexity of finding approximate equilibria [9].

Recent work has substantially improved the lower bounds. Edelsbrunner, Fillmore, and Oliveira constructed, for every integer $l \geq 1$, a configuration of $n_l = 8^l$ unit positive charges with exactly $\frac{25}{7}(n_l - 1)$ equilibria [5]. Arathoon, Ball, and Kvalheim then constructed five positive charges with at least 24 nondegenerate equilibria and showed that a small perturbation gives a finite Morse configuration. Their iteration yields, for every integer $m \geq 0$ and $n = 3 + 2m$, at least $4 + 20m = 10n - 26$ equilibria [2]. A separate positive cubature construction gives the superlinear asymptotic bound $M_+(n) \geq (3.596 \times 10^{-4} - o(1))n^{3/2}$ for positive charges [14]. The present result uses signed charges and a different mechanism to obtain the larger exponent 3.

1.4 Overview of our proof

Our argument first creates many nondegenerate critical points in explicit polynomial models and then realizes all of those models, on separated scales, within one Coulomb potential.

1. *Moment control on two rays.* For $t > 0$, the multipole expansion of a charge of strength q at tw converges for $|x| < t$. In degree l it contains only the solid zonal harmonic about w , with coefficient $qt^{-(l+1)}$. For d distinct sites on a ray, the map from the d strengths to the coefficients in degrees $1, \dots, d$ is a Vandermonde isomorphism. In the alternating jet through degree L , the highest even and odd degrees are

$$d_u = 2 \left\lfloor \frac{L}{2} \right\rfloor, \quad d_v = 2 \left\lceil \frac{L}{2} \right\rceil - 1.$$

We use d_u sites on the u -ray and d_v on the v -ray, for a total of $2L - 1$ sites. On each ray we prescribe the coefficients through its own highest required degree, including the intervening zeros. The remaining moments, including the degree- L moment on the shorter ray, are controlled by the rescaling and tail estimate described below. The constant term is immaterial to the critical-point equations.

2. *The two-degree local model.* For an integer $k \geq 2$ and $a, b \in \mathbb{R} \setminus \{0\}$, consider $aZ_k^u + bZ_{k+1}^v$. Write $r = |x|$, $\alpha = \langle x, u \rangle / r$, and $\beta = \langle x, v \rangle / r$. Outside the plane spanned by the axes, these are local coordinates on each of the two half-spaces, and the angular critical-point equations separate as $P'_k(\alpha) = P'_{k+1}(\beta) = 0$. A pair of roots produces two reflected critical points precisely when $\alpha^2 + \beta^2 < 1$ and

$$r = -\frac{k a P_k(\alpha)}{(k+1) b P_{k+1}(\beta)} > 0.$$

The Legendre values in this expression are nonzero because a Legendre polynomial and its derivative have no common zero. Simultaneous antipodal reflection of (α, β) partitions the g_k pairs satisfying the disk condition into two-element orbits and reverses the radial condition, so exactly one member of each orbit satisfies it. Each such member gives two critical points, exchanged by reflection across the axes' plane. The total is therefore $2(g_k/2) = g_k$. At each counted point, the coordinate Hessian is diagonal. The radial relation and Legendre's differential equation show that all three diagonal entries are nonzero. Congruence of Hessians at a critical point then gives nondegeneracy in Cartesian coordinates. There is also one explicit nondegenerate critical point on the axis of the odd-degree summand. This axial calculation applies to $k = 1$ as well, so the marked count on every rung is $g_k + 1$.

3. *Quadratic production at each rung.* Writing $\alpha = -\sin \varphi$ and $\beta = -\sin \psi$, with $\varphi, \psi \in (-\pi/2, \pi/2)$, converts the disk condition exactly into

$$|\varphi| + |\psi| < \frac{\pi}{2}.$$

Standard localization bounds for Legendre zeros place the roots of P'_k and P'_{k+1} on two nearly uniform angular meshes. Counting mesh points in this diamond gives

$$g_k = \frac{1}{2}k^2 + O(k), \quad S_L = \sum_{k=1}^{L-1} g_k = \frac{1}{6}L^3 + O(L^2).$$

The axial points increase this count to $B_L = S_L + L - 1$, leaving the leading coefficient unchanged. Since $L = \lceil n/2 \rceil = n/2 + O(1)$, the leading term $L^3/6$ gives the coefficient $1/48$ in the theorem.

4. *Scale separation and Coulomb realization.* For fixed L , choose the zonal axis alternately by degree and set

$$P_T = \sum_{l=1}^L T^{-l(l-1)/2} \zeta_l, \quad T > 1, \quad \zeta_l = \begin{cases} Z_l^u, & l \text{ even}, \\ Z_l^v, & l \text{ odd}. \end{cases}$$

For $1 \leq k < L$, homogeneity gives

$$T^{-k(k+1)/2} P_T(T^k Y) = \zeta_k(Y) + \zeta_{k+1}(Y) + O(T^{-1}) \quad \text{in } C^2$$

on a fixed annulus containing the selected local critical points. The limiting pair is the model above, with the axes interchanged when k is odd. Its selected critical points persist under this perturbation, and a sufficiently large T makes the corresponding annuli in the original coordinates pairwise disjoint.

Fix such a T . The two moment isomorphisms supply actual Coulomb potentials V_ε on $2L - 1$ fixed sites such that

$$V_\varepsilon(\varepsilon Y) - V_\varepsilon(0) \longrightarrow P_T(Y) \quad \text{in } C^2$$

on a fixed ball containing all the selected critical points of P_T . The uncontrolled moments contribute $O(\varepsilon)$ there; harmonic interior estimates give the required derivative bounds. Thus one first chooses T sufficiently large and then ε sufficiently small. The resulting Coulomb potential retains at least B_L nondegenerate equilibria.

5. *Morse completion, padding, and finite thresholds.* The moment conditions already force both signs. Small perturbations make every strength and the total charge nonzero while preserving both signs and the marked equilibria. A transversality argument in the site variables makes all equilibria nondegenerate; the final sites need not remain on the two rays. The nonzero monopole term excludes equilibria near infinity, while the singular term of each charge excludes them near its site. The critical set is therefore compact and, by nondegeneracy, discrete, hence finite. Adding sufficiently small nonzero charges at new sites and applying the same completion argument transfers the lower bound to every larger number of sites. Finally, exact rational root isolation for $2 \leq k \leq 26$ supplies the finite base cases, and the analytic lower bound for g_k propagates them to give the explicit thresholds $n = 41$ and $n \geq 43$.

1.5 Organization and scope

Section 2 proves the ray-moment isomorphism. Section 3 counts the off-plane critical points of the two-term zonal model, identifies an additional axial point, and proves both the asymptotic and finite estimates for g_k . Section 4 separates the models by scale and realizes the resulting harmonic jet by point charges. Section 5 proves finiteness, genericity, and padding and then completes the proof of Theorem 1.1. Section 6 explains why two axes are necessary for the off-axis two-degree mechanism used here. Section 7 records the exact finite-count verification and gives the location of the public verification archive.

Our cubic construction uses mixed signs, and the displayed finite estimates do not settle $n = 42$. The main theorem assumes no unproved conjecture. Its asymptotic coefficient $1/48$ and the existence, persistence, genericity, and finiteness arguments are proved analytically. The

explicit finite thresholds use the exact computation in Lemma 3.4, whose certificate is checked independently by integer and rational arithmetic as described in Section 7. We use throughout the standard facts that, for $l \geq 1$, $|P_l(t)| \leq 1$ on $[-1, 1]$, $P_l(-t) = (-1)^l P_l(t)$, and all zeros of P_l are simple and lie in $(-1, 1)$.

2 Ray moments

Lemma 2.1 (Ray expansion and moment isomorphism). *Let $w \in \mathbb{R}^3$ be a unit vector and let $t > 0$. For $|x| < t$,*

$$\frac{1}{|x - tw|} = \sum_{l=0}^{\infty} t^{-(l+1)} Z_l^w(x).$$

The series converges absolutely and uniformly on every closed ball centered at the origin of radius less than t .

More generally, fix an integer $L \geq 1$ and distinct numbers $t_1, \dots, t_L > 0$, put $\rho = \min_{1 \leq i \leq L} t_i$, and place real strengths $q_1, \dots, q_L$ at the sites $t_1 w, \dots, t_L w$. The strengths are allowed to vanish in this intermediate moment problem. For $|x| < \rho$,

$$U_q(x) := \sum_{i=1}^L \frac{q_i}{|x - t_i w|} = \sum_{l=0}^{\infty} c_l Z_l^w(x), \quad c_l = \sum_{i=1}^L q_i t_i^{-(l+1)};$$

this series also converges absolutely and uniformly on every closed ball centered at the origin of radius less than ρ . Thus its degree- l homogeneous term is $c_l Z_l^w$. For these fixed sites, the map

$$(q_1, \dots, q_L) \longmapsto (c_1, \dots, c_L)$$

is a linear isomorphism of $\mathbb{R}^L$.

Proof. For $x \neq 0$, put $r = |x|$ and $\mu = \langle x, w \rangle / r$. Since $|w| = 1$, the generating function for the Legendre polynomials gives

$$\frac{1}{|x - tw|} = \frac{1}{t} \left(1 - 2\frac{r}{t}\mu + \frac{r^2}{t^2} \right)^{-1/2} = \frac{1}{t} \sum_{l=0}^{\infty} P_l(\mu) \left(\frac{r}{t} \right)^l,$$

which is the asserted expansion. At $x = 0$ the identity follows from the polynomial extension in (4): $Z_0^w(0) = 1$, whereas $Z_l^w(0) = 0$ for $l \geq 1$. On every closed ball $|x| \leq R < t$, the bound $|P_l(\mu)| \leq 1$ majorizes the series by $t^{-1} \sum_{l \geq 0} (R/t)^l$. Thus the convergence is absolute and locally uniform in $|x| < t$. Applying this expansion at each of the finitely many sites and summing term by term proves the formula for the coefficients c_l . More explicitly, $|c_l| \leq (\sum_i |q_i|) \rho^{-(l+1)}$. For $0 < R < \rho$, one has $|Z_l^w(x)| \leq R^l$ on $|x| \leq R$. Hence, for every integer $m \geq 0$,

$$\sup_{|x| \leq R} \left| U_q(x) - \sum_{l=0}^m c_l Z_l^w(x) \right| \leq \frac{\sum_{i=1}^L |q_i|}{\rho} \frac{(R/\rho)^{m+1}}{1 - R/\rho}. \quad (5)$$

This proves the stated convergence for the finite charge sum as well.

Put $s_i = t_i^{-1}$. The matrix of the moment map, with row index $1 \leq l \leq L$ and column index $1 \leq i \leq L$, is

$$[t_i^{-(l+1)}] * l, i = 1^L = [s_i^{l-1}] * l, i = 1^L \text{diag}(s_1^2, \dots, s_L^2).$$

Its determinant is

$$\left(\prod_{i=1}^L s_i^2 \right) \prod_{1 \leq i < j \leq L} (s_j - s_i),$$

which is nonzero because the positive numbers s_i are distinct. Hence the moment map is a linear isomorphism. $\square$

Thus, with the L sites fixed, arbitrary real values of $c_1, \dots, c_L$ determine a unique real strength vector. They also determine $c_0 = U_q(0)$ and every c_l with $l > L$; these coefficients cannot be prescribed independently. The constant term does not affect critical points and is removed later by subtracting the value of the potential at the origin. The higher-degree terms are controlled by a tail estimate and spatial rescaling in Section 4. Nor does the lemma prescribe a general harmonic polynomial in any degree: it controls only the zonal component about the fixed axis w . This is exactly the restricted moment control used below.

3 The two-term model

We first count all critical points outside the plane spanned by the two axes, and then identify one additional nondegenerate point in that plane. These give the selected points used in the scale-separated construction; we do not need to classify the remaining in-plane critical points.

Lemma 3.1 (Perpendicular two-term model). *Let $k \geq 2$ be an integer, let $a, b \in \mathbb{R} \setminus \{0\}$, and set*

$$m = aZ_k^u + bZ_{k+1}^v, \quad u = e_3, \quad v = e_1.$$

The critical points of m outside the plane $\{x_2 = 0\}$ are in bijection with

$$\left\{ (\alpha, \beta) \in Z(P'_k) \times Z(P'_{k+1}) : \begin{array}{l} \alpha^2 + \beta^2 < 1, \\ \frac{aP_k(\alpha)}{bP_{k+1}(\beta)} < 0 \end{array} \right\} \times \{\pm 1\}.$$

For such a pair, the two points are

$$x = r(\beta, \pm \sqrt{1 - \alpha^2 - \beta^2}, \alpha), \quad r = -\frac{k a P_k(\alpha)}{(k+1) b P_{k+1}(\beta)}.$$

They are nondegenerate and have Morse index 1 or 2. The total number of these off-plane critical points is g_k , independently of the signs of a and b .

Proof. On the half-space $\{x_2 > 0\}$, the map

$$(r, w_1, w_3) \mapsto r(w_1, \sqrt{1 - w_1^2 - w_3^2}, w_3)$$

is a diffeomorphism from $(0, \infty) \times D$ onto $\{x_2 > 0\}$, where $D = \{(w_1, w_3) : w_1^2 + w_3^2 < 1\}$. Note first that m depends on x only through x_1, x_3 and $|x|$, hence only through x_1, x_3 and x_2^2 ; so m is invariant under the reflection $x_2 \mapsto -x_2$, and it suffices to analyze $\{x_2 > 0\}$. In these coordinates,

$$m = ar^k P_k(w_3) + br^{k+1} P_{k+1}(w_1).$$

The two angular derivatives are $ar^k P'_k(w_3)$ and $br^{k+1} P'_{k+1}(w_1)$; since $r > 0$ and $a, b \neq 0$, they vanish precisely when $P'_k(w_3) = P'_{k+1}(w_1) = 0$. Write $(w_3, w_1) = (\alpha, \beta)$. Membership in the chart requires $(\beta, \alpha) \in D$, which is exactly the disc condition $\alpha^2 + \beta^2 < 1$. The zeros of P_l are simple, so P_l and P'_l have no common zero and $P_k(\alpha), P_{k+1}(\beta)$ are nonzero. The radial derivative is $kar^{k-1} P_k(\alpha) + (k+1)br^k P_{k+1}(\beta)$, which vanishes exactly at the value r displayed above; that value lies in the chart, that is $r > 0$, exactly under the stated sign condition. Reflection across $\{x_2 = 0\}$ gives the second point.

At such a critical point use the coordinate order (r, w_3, w_1) and put

$$A = aP_k(\alpha), \quad B = bP_{k+1}(\beta).$$

The mixed angular derivative is identically zero, and the radial-angular derivatives vanish because $P'_k(\alpha) = P'_{k+1}(\beta) = 0$. Thus the Hessian of m in these coordinates is diagonal; using the radial equation in the form $kA = -(k+1)rB$ for the first entry, it equals

$$\text{diag}\left(-kr^{k-2}A, r^k aP''_k(\alpha), r^{k+1} bP''_{k+1}(\beta)\right).$$

At a zero $t \in (-1, 1)$ of P'_l , Legendre's differential equation gives

$$(1-t^2)P''_l(t) = -l(l+1)P_l(t),$$

so that

$$aP''_k(\alpha) = -\frac{k(k+1)}{1-\alpha^2}A, \quad bP''_{k+1}(\beta) = -\frac{(k+1)(k+2)}{1-\beta^2}B.$$

Since $r > 0$, the relation $kA = -(k+1)rB$ forces $\text{sgn } B = -\text{sgn } A$. The three diagonal signs are therefore

$$(-\text{sgn } A, -\text{sgn } A, +\text{sgn } A),$$

and none is zero. Thus the index is 2 when $A > 0$ and 1 when $A < 0$. A change of coordinates acts on the Hessian at a critical point by congruence, so by Sylvester's law of inertia this computation proves both nondegeneracy and the asserted Morse index in Cartesian coordinates.

It remains to count. Write

$$G_k = \{(\alpha, \beta) \in Z(P'_k) \times Z(P'_{k+1}) : \alpha^2 + \beta^2 < 1\},$$

so that $\#G_k = g_k$, and let $\sigma(\alpha, \beta) = (-\alpha, -\beta)$. Since $P_l(-t) = (-1)^l P_l(t)$, we have $P'_l(-t) = (-1)^{l+1} P'_l(t)$, so $Z(P'_k)$ and $Z(P'_{k+1})$ are symmetric about 0; as σ also preserves the disc condition, σ maps G_k to itself. Because P_k and P_{k+1} have opposite parity, the ratio $aP_k(\alpha)/(bP_{k+1}(\beta))$ changes sign under σ . That ratio is finite and nonzero, so it cannot equal its own negative; hence σ has no fixed point in G_k . (Equally directly: a fixed point would be $(0, 0)$, and $P'_l(0) = 0$ exactly when l is even, so P'_k and P'_{k+1} cannot both vanish at 0.) Therefore σ splits G_k into $g_k/2$ orbits of size two, and exactly one member of each orbit satisfies the sign condition. The reflection $x_2 \mapsto -x_2$ restores the factor 2, so the number of off-plane critical points is exactly g_k . $\square$

Lemma 3.2 (An additional axial point). *Let $k \geq 1$ be an integer, let $a, b \in \mathbb{R} \setminus \{0\}$, and let u, v be perpendicular unit vectors. The polynomial*

$$m = aZ_k^u + bZ_{k+1}^v$$

has a nonzero nondegenerate critical point on the axis of its odd-degree summand. This point is distinct from the off-plane points counted in Lemma 3.1.

Proof. Choose a unit vector w perpendicular to u and v . We use signed coordinates $t \neq 0$ on each axis. We first record the gradient and Hessian identities in orthonormal frames. Let (h, e, f) be an orthonormal basis. For every integer $l \geq 1$, one has $Z_l^h(th) = t^l$, including negative t , by the parity of P_l . Rotational symmetry about h therefore gives

$$\nabla Z_l^h(th) = lt^{l-1}h, \quad D^2 Z_l^h(th) = l(l-1)t^{l-2} \text{diag}(1, -1/2, -1/2)$$

in the basis (h, e, f) . Indeed, the two transverse Hessian entries are equal, the mixed entries vanish, and harmonicity makes the trace zero.

For even $l \geq 2$, put $c_l = P_l(0)$. On the equatorial plane spanned by e, f , the polynomial is $c_l(x_e^2 + x_f^2)^{l/2}$, where x_e, x_f are the coordinates in that plane. It is even in the h -coordinate. Consequently,

$$\nabla Z_l^h(te) = lc_l t^{l-1}e, \quad D^2 Z_l^h(te) = lc_l t^{l-2} \text{diag}(l-1, 1, -l)$$

in the basis (e, f, h) . The first two diagonal entries follow by differentiating the equatorial restriction; the mixed entries vanish by reflection symmetry, and the last diagonal entry follows from the zero trace. These formulas hold for both signs of t . Recall also that

$$P_{2j}(0) = (-1)^j \frac{\binom{2j}{j}}{4^j} \neq 0 \quad (j = 0, 1, 2, \dots).$$

On the axis of the odd-degree summand, both summand gradients are parallel to that axis. Thus a critical point of the restriction to that axis is a critical point in $\mathbb{R}^3$.

If k is even, put $c = P_k(0) \neq 0$. On the axis tv the potential is $act^k + bt^{k+1}$, and its unique nonzero critical parameter is

$$t = -\frac{akc}{b(k+1)}.$$

With $D = akct^{k-2} \neq 0$, substitution in the Hessian formulas gives

$$D^2 m(tv) = D \text{diag} \left(-1, \frac{k+2}{2}, -\frac{k}{2} \right)$$

in the basis (v, w, u) . Every eigenvalue is nonzero.

If k is odd, put $c = P_{k+1}(0) \neq 0$. On the axis tu the potential is $at^k + bct^{k+1}$, with nonzero critical parameter

$$t = -\frac{ak}{b(k+1)c}.$$

Now $D = akt^{k-2} \neq 0$, and in the basis (u, w, v) ,

$$D^2 m(tu) = D \text{diag} \left(-1, -\frac{k+1}{2}, \frac{k+3}{2} \right).$$

Again every eigenvalue is nonzero. The case $k = 1$ is included: $c = -1/2$, $t = a/b$, and the Hessian is $b \text{diag}(-1, -1, 2)$. The point always lies in the axes' plane and therefore cannot coincide with any of the off-plane points. $\square$

Together, Lemmas 3.1 and 3.2 supply $g_k + 1$ distinct nonzero nondegenerate critical points for every $k \geq 2$. For $k = 1$, the axial lemma supplies the single required point, since $g_1 = 0$. An orthogonal change of coordinates preserves these conclusions, in particular when the two perpendicular axes are interchanged.

Lemma 3.3 (Quadratic count). *For every integer $k \geq 3$,*

$$\frac{k^2}{2} - 5k - 6 \leq g_k \leq \frac{k^2}{2} + 6k + 19.$$

Consequently, $g_k = \frac{1}{2}k^2 + O(k)$ as $k \rightarrow \infty$.

Proof. Fix an integer $k \geq 3$ and put $m = k + \frac{1}{2}$ and $m' = k + \frac{3}{2}$. Write $\alpha = \cos \theta = -\sin \varphi$, where $\varphi = \theta - \pi/2 \in (-\pi/2, \pi/2)$, and similarly write $\beta = -\sin \psi$ with $\psi \in (-\pi/2, \pi/2)$. Indeed, if $a = |\varphi|$ and $b = |\psi|$, then $a, b \in [0, \pi/2)$, and monotonicity of $\sin$ on $[0, \pi/2]$ gives $\sin^2 a + \sin^2 b < 1$ if and only if $\sin b < \cos a = \sin(\pi/2 - a)$, or equivalently $a + b < \pi/2$. Hence

$$\alpha^2 + \beta^2 < 1 \iff |\varphi| + |\psi| < \frac{\pi}{2}. \quad (6)$$

Thus the admissible angular region is the open diamond of area $\pi^2/2$.

Let $0 < \theta_1 < \dots < \theta_k < \pi$ be the zeros of $P_k(\cos \theta)$. A standard localization of the Legendre zeros [20, Theorem 6.21.2] states that

$$\left(j - \frac{1}{2}\right) \frac{\pi}{m} \leq \theta_j \leq j \frac{\pi}{m}, \quad 1 \leq j \leq k.$$

Rolle's theorem gives angles $\xi_j \in (\theta_j, \theta_{j+1})$ satisfying $P'_k(\cos \xi_j) = 0$, for $j = 1, \dots, k-1$; since $\deg P'_k = k-1$, these are all the zeros of P'_k in angular coordinates and each is simple. The inclusion $\xi_j \in (\theta_j, \theta_{j+1})$ is strict, so

$$\left(j - \frac{1}{2}\right) \frac{\pi}{m} < \xi_j < (j+1) \frac{\pi}{m}$$

holds strictly even though the cited localization was stated with $\leq$. Since $\pi/2 = (\pi/m)(k/2 + 1/4)$, the angles $\varphi_j = \xi_j - \pi/2$ obey the *two-sided* estimate

$$\left|\varphi_j - \frac{\pi}{m} p_j\right| < \frac{3\pi}{4m}, \quad p_j := j - \frac{k}{2}, \quad 1 \leq j \leq k-1. \quad (7)$$

The zeros ψ_i of P'_{k+1} , written in the same way, obey

$$\left|\psi_i - \frac{\pi}{m'} q_i\right| < \frac{3\pi}{4m'}, \quad q_i := i - \frac{k+1}{2}, \quad 1 \leq i \leq k. \quad (8)$$

By (6) and the upper halves of (7)–(8), the pair (j, i) is admissible whenever

$$\frac{|p_j|}{m} + \frac{|q_i|}{m'} < \tau, \quad \tau = \frac{1}{2} - \frac{3}{4} \left(\frac{1}{m} + \frac{1}{m'} \right). \quad (9)$$

We now count this sufficient set. Put $\Lambda_p = p_1 + \mathbb{Z}$ and $\Lambda_q = q_1 + \mathbb{Z}$. As j runs from 1 to $k-1$, the values p_j are precisely

$$\Lambda_p \cap \left[-\left(\frac{k}{2} - 1\right), \frac{k}{2} - 1 \right],$$

and as i runs from 1 to k , the values q_i are precisely

$$\Lambda_q \cap \left[-\frac{k-1}{2}, \frac{k-1}{2} \right].$$

Thus Λ_p and Λ_q are step-1 lattices symmetric about 0. The displayed index ranges impose no further restriction: every pair $(p, q) \in \Lambda_p \times \Lambda_q$ satisfying (9) does come from indices $1 \leq j \leq k-1$ and $1 \leq i \leq k$, since (9) forces $|p| < m\tau$ and $|q| < m'\tau$, while

$$m\tau = \frac{k}{2} - \frac{1}{2} - \frac{3m}{4m'} < \frac{k}{2} - 1, \quad m'\tau = \frac{k}{2} - \frac{3m'}{4m} < \frac{k-1}{2}.$$

The first strict inequality uses $m/m' > 2/3$, and the second uses $m'/m > 1$; both hold for every $k \geq 2$. For fixed $p \in \Lambda_p$ with $|p| < m\tau$, the admissible q lie in an interval of length $2m'(\tau - |p|/m)$. An open interval of length ℓ contains at least $\ell - 1$ points of any step-1 lattice. Restrict to $|p| \leq R := m\tau - 1$, which is nonnegative for $k \geq 5$. There are at least $2R - 1$ such p 's. Also $m'\tau > m\tau \geq 1$, so $2m'\tau - 1 > 0$; this positivity is what lets the (possibly negative) bound $2R - 1$ be substituted for the number of such p 's below. Moreover,

$$\sum_{\substack{p \in \Lambda_p \\ |p| \leq R}} |p| \leq R^2 + R + \frac{1}{4}. \quad (10)$$

Indeed, for the integer lattice the left side is $h(h+1)$, where $h = \lfloor R \rfloor$; for the half-integer lattice it is $(h+1)^2$, where $h+1/2$ is the largest nonnegative lattice point not exceeding R . If the half-integer set is empty, the left side is instead 0. All three cases give (10).

Let $N(k)$ denote the number of lattice pairs satisfying (9). Summing the interval bound over the values $|p| \leq R$ alone, and discarding the remaining p 's, gives

$$\begin{aligned} N(k) &\geq (2R - 1)(2m'\tau - 1) - \frac{2m'}{m} \left(R^2 + R + \frac{1}{4} \right) \\ &= 2mm'\tau^2 - 2m\tau - 4m'\tau + 3 - \frac{m'}{2m}. \end{aligned}$$

For $k \geq 5$, both τ and $\frac{1}{2} - \frac{3}{2m}$ are positive. Since $\tau > \frac{1}{2} - \frac{3}{2m}$ and $m'/(2m) < 1$, we obtain

$$2mm'\tau^2 \geq \frac{mm'}{2} - 3m', \quad 2m\tau + 4m'\tau < m + 2m' = 3k + \frac{7}{2}.$$

Substituting $\frac{mm'}{2} = \frac{k^2}{2} + k + \frac{3}{8}$ and $3m' = 3k + \frac{9}{2}$ therefore gives

$$N(k) > \left(\frac{k^2}{2} + k + \frac{3}{8} \right) - \left(3k + \frac{9}{2} \right) - \left(3k + \frac{7}{2} \right) + 3 - 1 = \frac{k^2}{2} - 5k - \frac{45}{8}.$$

Since (9) is sufficient for admissibility and distinct lattice pairs come from distinct index pairs, $g_k \geq N(k)$, so

$$g_k > \frac{k^2}{2} - 5k - \frac{45}{8} > \frac{k^2}{2} - 5k - 6 \quad (k \geq 5).$$

For $k = 3, 4$ the asserted lower bound is negative and hence automatic.

For the asymptotic a matching upper bound is needed. By (6) and the *lower* halves of (7)–(8), every admissible pair satisfies the necessary condition

$$\frac{|p_j|}{m} + \frac{|q_i|}{m'} < \tau^+, \quad \tau^+ = \frac{1}{2} + \frac{3}{4} \left(\frac{1}{m} + \frac{1}{m'} \right). \quad (11)$$

Distinct index pairs give distinct lattice pairs, so g_k is at most the number of pairs $(p, q) \in \Lambda_p \times \Lambda_q$ obeying (11). Assign to each such pair the closed unit square centered at it. Both lattices have step 1, so these squares have pairwise disjoint interiors. For every point (η, ζ) in the square centered at (p, q) , the weighted norm increases by at most $1/(2m) + 1/(2m')$. Consequently, each square is contained in the diamond

$$\Delta = \left\{ (\eta, \zeta) : \frac{|\eta|}{m} + \frac{|\zeta|}{m'} \leq \tau^{++} \right\}, \quad \tau^{++} = \tau^+ + \frac{1}{2m} + \frac{1}{2m'} = \frac{1}{2} + \frac{5}{4} \left(\frac{1}{m} + \frac{1}{m'} \right),$$

whose area is $2mm'(\tau^{++})^2$. Comparing total areas,

$$g_k \leq 2mm'(\tau^{++})^2 = \frac{mm'}{2} + \frac{5}{2}(m + m') + \frac{25}{8} \cdot \frac{(m + m')^2}{mm'}.$$

Here $mm' = k^2 + 2k + \frac{3}{4}$ and $m + m' = 2k + 2$, so the first two terms are $\frac{1}{2}k^2 + 6k + \frac{43}{8}$, while the third equals $\frac{25}{8} \cdot 4(k+1)^2 / ((k+1)^2 - \frac{1}{4})$, which decreases in k and is therefore at most its value $\frac{25}{8} \cdot \frac{144}{35} = \frac{90}{7} < 13$ at $k = 2$. Hence

$$\frac{k^2}{2} - 5k - 6 \leq g_k \leq \frac{k^2}{2} + 6k + 19 \quad (k \geq 3),$$

so $g_k = \frac{1}{2}k^2 + O(k)$; in particular $g_k = (\frac{1}{2} + o(1))k^2$. This proves the assertion. The constant $\frac{1}{2}$ is the ratio of the area of the diamond (6) to that of the square $(-\pi/2, \pi/2)^2$ swept out by the two angular meshes. $\square$

Lemma 3.4 (Exact finite counts). *The values needed for the finite thresholds in Theorem 1.1 are*

$$\begin{aligned} (g_2, \dots, g_{14}) &= (2, 6, 8, 16, 22, 30, 36, 48, 58, 70, 80, 96, 110), \\ (g_{15}, \dots, g_{26}) &= (126, 140, 160, 178, 198, 216, 240, 262, 286, 308, 336, 362). \end{aligned}$$

In particular,

$$S_{21} = 1600, \quad S_{22} = 1840, \quad S_{23} = 2102, \quad S_{24} = 2388.$$

Proof. This is a finite exact computation. For each $2 \leq k \leq 26$, let

$$R_k := \text{Res} * \beta(P * k + 1'(\beta), \text{Res}_\alpha(P'_k(\alpha), \alpha^2 + \beta^2 - 1)).$$

For each of P'_k and P'_{k+1} , exact real-root isolation gives pairwise disjoint rational intervals, each containing exactly one simple root and together accounting for every root. If a pair of these zeros satisfied $\alpha^2 + \beta^2 = 1$, then the inner resultant would vanish at β , forcing $R_k = 0$. The exact computation certifies $R_k \neq 0$; therefore no root pair lies on the boundary circle. There are only finitely many root pairs, so rational interval refinement eventually proves, for each one, exactly one of the strict inequalities $\alpha^2 + \beta^2 < 1$ and $\alpha^2 + \beta^2 > 1$.

The ancillary file `verify_counts.py` implements these steps using exact rational polynomial and interval arithmetic, verifies all root multiplicities and inequalities, and prints the values above and their partial sums; the command to reproduce the verification appears in Section 7. It has been tested with SymPy 1.14.0 [21]; the accompanying JSON certificate records the version used and all isolating intervals and decisions. The independent checker `check_count_certificate.py` reconstructs the polynomials, certifies the intervals by rational Sturm counts, and rechecks the resultants and all strict inequalities without using SymPy. Thus no count is accepted on the basis of a floating-point approximation or a numerical tolerance. The complete verification workflow is described in Section 7. $\square$

4 The scale-separated chain

We first combine the two-term models into one harmonic polynomial, with their selected critical points at disjoint spatial scales. We then fix that polynomial and approximate it, after rescaling, by a Coulomb potential. Only the selected nondegenerate points are tracked; finiteness and nondegeneracy of the entire critical set will be arranged in Section 5.

We use the following elementary persistence fact in both steps. Suppose f is a real C^2 function near p , with $\nabla f(p) = 0$ and $H = D^2 f(p)$ invertible. Choose a sufficiently small closed ball $K = \{x : |x - p| \leq r\}$ on which $\|I - H^{-1}D^2 f(x)\| \leq 1/4$, where the norm is the Euclidean operator norm. Every function g sufficiently close to f in $C^2(K)$ then satisfies

$$\sup_{x \in K} \|I - H^{-1}D^2 g(x)\| \leq \frac{1}{2}, \quad |H^{-1}\nabla g(p)| < \frac{r}{2}.$$

Thus $x \mapsto x - H^{-1}\nabla g(x)$ is a contraction taking K into its interior. Its unique fixed point is a critical point of g , and the first inequality implies that $D^2 g$ is invertible there. For finitely many distinct nondegenerate critical points, choose these balls pairwise disjoint and take a common perturbation bound.

Lemma 4.1 (Scale separation). *Let $L \geq 2$ be an integer. For $1 \leq l \leq L$, let ζ_l be a real harmonic homogeneous polynomial of degree l on $\mathbb{R}^3$, and define*

$$P_T = \sum_{l=1}^L T^{-l(l-1)/2} \zeta_l, \quad T > 1.$$

Suppose that, for each $1 \leq k < L$, a specified finite set of N_k distinct nondegenerate critical points of $\zeta_k + \zeta_{k+1}$ lies in the annulus $W^{-1} \leq |Y| \leq W$, where $W \geq 1$. Then, for all sufficiently large T , the polynomial P_T has at least $\sum_{k=1}^{L-1} N_k$ nondegenerate critical points.

Proof. For each $1 \leq k < L$, define the rescaled polynomial

$$F_{T,k}(Y) := T^{-k(k+1)/2} P_T(T^k Y).$$

Homogeneity gives

$$\begin{aligned} F_{T,k}(Y) &= \zeta_k(Y) + \zeta_{k+1}(Y) \\ &\quad + \sum_{\substack{1 \leq l \leq L \\ l \neq k, k+1}} T^{-(l-k)(l-k-1)/2} \zeta_l(Y). \end{aligned}$$

For $l \neq k, k+1$, the integer $l-k$ is neither 0 nor 1, so the exponent $(l-k)(l-k-1)/2$ is at least 1. Therefore the final sum is $O(T^{-1})$ in C^2 on the compact annulus

$$A := \{Y : (2W)^{-1} \leq |Y| \leq 2W\}.$$

The original annulus lies in the interior of A . Around the N_k selected points choose pairwise disjoint balls with closures in the interior of A , small enough for the persistence argument above. For all sufficiently large T , each ball therefore contains a nondegenerate critical point of $F_{T,k}$. In particular, these points are distinct and lie in A .

If $x = T^k Y$, the chain rule gives

$$\nabla_Y F_{T,k}(Y) = T^{-k(k+1)/2+k} \nabla P_T(x), \quad D_Y^2 F_{T,k}(Y) = T^{-k(k+1)/2+2k} D^2 P_T(x).$$

Both prefactors are positive. Thus the corresponding points are distinct nondegenerate critical points of P_T , and in the original variable they lie in

$$\frac{T^k}{2W} \leq |x| \leq 2WT^k.$$

The finitely many values of k admit a common lower bound on T . Enlarge that bound, if necessary, so that $T > (2W)^2$; then

$$2WT^k < \frac{T^{k+1}}{2W}.$$

Thus each displayed annulus lies strictly below the next one. Hence no critical point is counted at two scales, and summing over k proves the result. $\square$

Apply the lemma to

$$\zeta_l = \begin{cases} Z_l^u, & l \text{ even,} \\ Z_l^v, & l \text{ odd.} \end{cases} \quad (12)$$

For $k \geq 2$ even, $\zeta_k + \zeta_{k+1} = Z_k^u + Z_{k+1}^v$ and Lemmas 3.1 and 3.2 with $a = b = 1$ supply $N_k = g_k + 1$ nondegenerate points. For $k \geq 3$ odd, $\zeta_k + \zeta_{k+1} = Z_k^v + Z_{k+1}^u$; let σ be the orthogonal involution that exchanges e_1 and e_3 . Then composition with σ turns this polynomial into $Z_k^u + Z_{k+1}^v$, because $Z_l^w \circ \sigma = Z_l^{\sigma w}$. The map σ preserves radius and carries critical points and Hessians by orthogonal changes of coordinates. The same two lemmas therefore again supply $N_k = g_k + 1$ nondegenerate points. For $k = 1$, Lemma 3.2 supplies $N_1 = 1 = g_1 + 1$. For fixed L , the union of all selected sets is finite and every selected point has positive radius, so one common $W \geq 1$ contains them in the annulus $W^{-1} \leq |Y| \leq W$. Lemma 4.1 consequently gives

$$\# \text{Crit}(P_T) \geq B_L = S_L + L - 1 \quad (13)$$

for all sufficiently large T , with at least B_L distinct nonzero nondegenerate points available to mark. This does not assert that $\text{Crit}(P_T)$ is finite or that its other points are nondegenerate.

Lemma 4.2 (Coulomb realization on two rays). *Fix an integer $L \geq 2$ and choose a sufficiently large T , together with B_L marked nondegenerate critical points of P_T furnished above. Put*

$$d_u = 2 \left\lfloor \frac{L}{2} \right\rfloor, \quad d_v = 2 \left\lceil \frac{L}{2} \right\rceil - 1.$$

Fix distinct $t_1, \dots, t_{d_u} \in [1, 2]$ and distinct $s_1, \dots, s_{d_v} \in [1, 2]$. The $d_u + d_v = 2L - 1$ sites $t_i u$ and $s_i v$ are pairwise distinct. There is a family $(V_\varepsilon)_{0 < \varepsilon \leq 1}$ of two-ray Coulomb sums with these fixed sites, allowing zero strengths at this intermediate stage, such that

$$Y \mapsto V_\varepsilon(\varepsilon Y) - V_\varepsilon(0)$$

converges to P_T in C^2 on every fixed ball centered at the origin as $\varepsilon \downarrow 0$; the rescaled potential is defined on that ball for all sufficiently small ε . In particular, a fixed ball may be chosen to

contain all B_L marked points. Hence, for all sufficiently small positive ε , the function V_ε has at least B_L nondegenerate critical points. For $L \geq 3$ and every $0 < \varepsilon \leq 1$, the strengths on each ray include at least one positive and at least one negative value; no nonvanishing assumption on the other strengths is needed at this stage. For $L = 2$, the full configuration still includes both signs.

Proof. Throughout this proof, L , T , the sites, and the marked points are fixed before ε tends to zero. No estimate uniform in a simultaneously varying T is needed. For $1 \leq l \leq d_u$, prescribe the moments on the u -ray by

$$c_l = \begin{cases} \varepsilon^{-l} T^{-l(l-1)/2}, & l \text{ even,} \\ 0, & l \text{ odd,} \end{cases}$$

and, for $1 \leq l \leq d_v$, prescribe the moments on the v -ray by

$$d_l = \begin{cases} 0, & l \text{ even,} \\ \varepsilon^{-l} T^{-l(l-1)/2}, & l \text{ odd.} \end{cases}$$

Lemma 2.1 gives unique charge vectors

$$q(\varepsilon) = (q_1(\varepsilon), \dots, q_{d_u}(\varepsilon)), \quad p(\varepsilon) = (p_1(\varepsilon), \dots, p_{d_v}(\varepsilon))$$

on the u -ray and the v -ray, respectively. For $L \geq 3$, these moment conditions force both signs on each ray. Indeed, $c_2 \neq 0$ shows that $q(\varepsilon)$ is not the zero vector, whereas $c_1 = \sum_i q_i(\varepsilon) t_i^{-2} = 0$. Because every weight t_i^{-2} is positive, the nonzero entries of $q(\varepsilon)$ cannot all have the same sign. Likewise, $d_1 \neq 0$ shows that $p(\varepsilon)$ is not the zero vector, whereas $d_v \geq 2$ and $d_2 = \sum_i p_i(\varepsilon) s_i^{-3} = 0$; hence $p(\varepsilon)$ also has both positive and negative entries. When $L = 2$, one has $(d_u, d_v) = (2, 1)$; the preceding argument still forces both signs on the u -ray, which is enough for the full configuration to use both signs. This is the precise point at which the mixed-sign hypothesis enters the construction. Define

$$V_\varepsilon(x) = \sum_{i=1}^{d_u} \frac{q_i(\varepsilon)}{|x - t_i u|} + \sum_{i=1}^{d_v} \frac{p_i(\varepsilon)}{|x - s_i v|}.$$

Put

$$\rho = \min(\{t_i : 1 \leq i \leq d_u\} \cup \{s_i : 1 \leq i \leq d_v\}) \geq 1.$$

The ball $\{|x| < \rho\}$ contains no site, and the Taylor series from Lemma 2.1 converges there. Homogeneity and the prescribed moments give

$$V_\varepsilon(\varepsilon Y) - V_\varepsilon(0) = P_T(Y) + R_\varepsilon(Y), \quad R_\varepsilon(Y) = \sum_{l > d_u} \varepsilon^l \tilde{c}_l(\varepsilon) Z_l^u(Y) + \sum_{l > d_v} \varepsilon^l \tilde{d}_l(\varepsilon) Z_l^v(Y),$$

where $\tilde{c}_l(\varepsilon)$ and $\tilde{d}_l(\varepsilon)$ are the actual ray moments. The prescribed terms sum exactly to P_T : its even-degree terms occur on the u -ray through degree d_u , and its odd-degree terms on the v -ray through degree d_v ; the other coefficients within these prescribed ranges are zero. On the shorter ray, which has $L - 1$ sites, the degree- L coefficient is not prescribed and need not vanish. It is part of R_ε , along with all subsequent terms on that ray. Subtraction of $V_\varepsilon(0)$ removes the uncontrolled degree-zero term.

We prove the required C^2 convergence. The two inverse moment maps are fixed after the sites are chosen, and $T^{-l(l-1)/2} \leq 1$ because $T > 1$. For either ray, write d for its number of

sites and $w(\varepsilon)$ for its strength vector. Every prescribed moment on that ray has absolute value at most ε^{-d} when $0 < \varepsilon \leq 1$. Boundedness of its fixed inverse linear map therefore gives

$$\|w(\varepsilon)\|_1 \leq K\varepsilon^{-d}$$

for a constant K depending on L and the chosen sites. Since

$$|\tilde{c}_l(\varepsilon)| \leq \|q(\varepsilon)\|_1 \rho^{-(l+1)}, \quad |\tilde{d}_l(\varepsilon)| \leq \|p(\varepsilon)\| * 1 \rho^{-(l+1)},$$

the bound $|Z_l^h(Y)| \leq |Y|^l$ for $h \in \{u, v\}$ shows that the absolute value of the remainder contribution from a ray with d sites is bounded, whenever $\varepsilon|Y| \leq \rho/2$, by

$$K\varepsilon^{-d} \rho^{-1} \sum *l > d \left(\frac{\varepsilon|Y|}{\rho} \right)^l \leq 2K\varepsilon^{-d} \rho^{-1} \left(\frac{\varepsilon|Y|}{\rho} \right)^{d+1}.$$

On a fixed ball $|Y| \leq R$, the last expression is at most $2KR^{d+1}\rho^{-(d+2)}\varepsilon$. Summing over the two rays therefore gives $R_\varepsilon = O(\varepsilon)$ uniformly on every fixed ball. It is important here to use each ray's own cutoff d : its strengths are $O(\varepsilon^{-d})$, while its first unprescribed term starts at degree $d+1$.

For $R > 0$, write $B_R = \{Y : |Y| \leq R\}$. Choose $R_0 \geq 1$ so that all the critical points of P_T selected in Lemma 4.1 lie in the interior of B_{R_0} . For sufficiently small ε , the preceding estimate applies on B_{2R_0} and gives $\|R_\varepsilon\|_{C^0(B_{2R_0})} = O(\varepsilon)$. Moreover, R_ε is harmonic on a neighborhood of B_{2R_0} , because the rescaled Coulomb potential has no site there and P_T is harmonic. The interior derivative estimates for harmonic functions [3, Theorem 2.4] give, for every multi-index α with $|\alpha| \leq 2$,

$$\sup_{B_{R_0}} |D^\alpha R_\varepsilon| \leq C_\alpha R_0^{-|\alpha|} \|R_\varepsilon\| * C^0(B * 2R_0).$$

Indeed, apply the estimate on the ball of radius R_0 about each point of B_{R_0} ; each such ball lies in B_{2R_0} . The constants are independent of ε , and R_0 is fixed, so

$$\|R_\varepsilon\| * C^2(B * R_0) = O(\varepsilon).$$

Every selected critical point of P_T is nondegenerate. The persistence argument at the start of this section, applied in pairwise disjoint balls with closures in the interior of B_{R_0} , therefore preserves all B_L points as distinct nondegenerate critical points of $P_T + R_\varepsilon$ for all sufficiently small positive ε . Finally,

$$\nabla_Y (V_\varepsilon(\varepsilon Y) - V_\varepsilon(0)) = \varepsilon \nabla V_\varepsilon(\varepsilon Y), \quad D_Y^2 (V_\varepsilon(\varepsilon Y) - V_\varepsilon(0)) = \varepsilon^2 D^2 V_\varepsilon(\varepsilon Y).$$

Thus rescaling $x = \varepsilon Y$ preserves critical points and nondegeneracy. It places the selected points in $\{|x| \leq \varepsilon R_0\} \subset \{|x| < \rho\}$, which contains no site. This proves the lemma. $\square$

5 Finiteness, genericity, and padding

The two-ray potential constructed above already has the required marked nondegenerate equilibria. We now arrange that every strength and the total charge are nonzero and that the entire critical set is finite and nondegenerate, while retaining nearby continuations of all marked points. The completion argument permits zero strengths initially and also allows additional sites, so it applies directly to the preceding construction and supplies the required padding to larger numbers of charges.

Lemma 5.1 (Finiteness). *Let $n \geq 1$ be an integer, let $a_1, \dots, a_n \in \mathbb{R}^3$ be distinct, let $q_1, \dots, q_n \in \mathbb{R} \setminus \{0\}$, and set*

$$V_{a,q}(x) = \sum_{i=1}^n \frac{q_i}{|x - a_i|}, \quad \Omega_a = \mathbb{R}^3 \setminus \{a_1, \dots, a_n\}.$$

If the total charge $Q = \sum_i q_i$ is nonzero and every critical point of $V_{a,q}$ is nondegenerate, then $\text{Crit}(V_{a,q})$ is finite.

Proof. Fix a site a_i and choose a closed ball centered at it that is disjoint from all other sites. The field due to the other charges is continuous on this compact ball and hence bounded there, say by C_i . On its punctured interior, the reverse triangle inequality gives

$$|\nabla V_{a,q}(x)| \geq \frac{|q_i|}{|x - a_i|^2} - C_i.$$

Because $q_i \neq 0$, this is positive in a sufficiently small punctured ball $0 < |x - a_i| < \delta_i$. Doing this for every site excludes a neighborhood of each singularity.

At infinity, for each fixed site a_i and uniformly in the direction of x ,

$$\frac{x - a_i}{|x - a_i|^3} = \frac{x}{|x|^3} + O(|x|^{-3}).$$

To verify the uniformity, set $G(y) = y/|y|^3$ for $y \neq 0$. Its derivative has operator norm $\|DG(y)\| = 2|y|^{-3}$. If $|x| \geq \max\{1, 2 \max_i |a_i|\}$, then $|x - sa_i| \geq |x|/2$ for $0 \leq s \leq 1$, so integration along this segment gives

$$|G(x - a_i) - G(x)| \leq 16|a_i| |x|^{-3}.$$

Summing the corresponding gradient formulas gives

$$\nabla V_{a,q}(x) = -\frac{Qx}{|x|^3} + O(|x|^{-3}).$$

The leading term has magnitude $|Q|/|x|^2$, so, because $Q \neq 0$, it strictly dominates the remainder for all sufficiently large $|x|$. Hence the field is nonzero outside a sufficiently large ball. Together with the preceding site exclusions, this places $\text{Crit}(V_{a,q})$ in the compact set

$$C = \{x : |x| \leq R, |x - a_i| \geq \delta_i \text{ for every } i\} \subset \Omega_a$$

for a sufficiently large R .

If $\text{Crit}(V_{a,q})$ were infinite, compactness would give an accumulation point $x_* \in C$. Continuity of the gradient would imply $\nabla V_{a,q}(x_*) = 0$, so x_* would be a nonisolated critical point. This contradicts the inverse function theorem, since the gradient has invertible derivative at every nondegenerate critical point. $\square$

Lemma 5.2 (Morse completion and padding). *Let $n \geq 1$ be an integer. Suppose a Coulomb sum with n distinct sites and arbitrary real charge strengths, possibly zero, has a finite set of K specified distinct nondegenerate critical points outside all the listed sites. For every integer $m \geq n$, there is a configuration of m distinct sites and nonzero real charges, with nonzero total charge and a finite, entirely nondegenerate critical set containing at least K points. Every originally nonzero strength can be chosen to retain its sign.*

Proof. Choose pairwise disjoint persistence neighborhoods $U_1, \dots, U_K$ of the marked points, with compact closures disjoint from the original sites. Choose $m - n$ additional sites outside these closures, all distinct from the original sites and from one another; when $m = n$ there is nothing to add. Extend the original strength vector by assigning strength zero to each new site, and denote the extended vector by $q^0 \in \mathbb{R}^m$. These zero-strength sites do not change the potential on the persistence neighborhoods. Strength vectors for which some entry is zero or the total strength is zero form a finite union of proper hyperplanes in $\mathbb{R}^m$. Its complement is dense, so choose a nearby strength vector $q = (q_1, \dots, q_m)$ such that every q_i and the total charge $Q = \sum_i q_i$ are nonzero. We may also require $|q_i - q_i^0| < |q_i^0|/2$ whenever $q_i^0 \neq 0$, preserving all original nonzero signs. Because all sites are a positive distance from the compact sets $\overline{U}_j$, this change of strengths can be chosen to be C^2 -small on their union. The K nondegenerate critical points therefore persist, one in each U_j , by the persistence argument in Section 4. Denote the resulting ordered site configuration by a^0 .

Fix this charge vector and vary the sites. Let $\mathcal{A}$ be the open configuration space of ordered, pairwise distinct m -tuples in $(\mathbb{R}^3)^m$, and define

$$\mathcal{E} = \{(a, x) : a \in \mathcal{A}, x \neq a_i \text{ for every } i\}, \quad F(a, x) = \nabla_x \sum_{i=1}^m \frac{q_i}{|x - a_i|}.$$

The set $\mathcal{E}$ is open in $\mathbb{R}^{3m+3}$, and F is smooth there. For each i ,

$$D_{a_i} F(a, x) = -q_i \operatorname{Hess}_x \left(\frac{1}{|x - a_i|} \right).$$

Explicitly, for $z = x - a_i \neq 0$,

$$\operatorname{Hess}_x \left(\frac{1}{|x - a_i|} \right) = \frac{3zz^\top - |z|^2 I}{|z|^5}.$$

Its radial and two transverse eigenvalues are $2|x - a_i|^{-3}$, $-|x - a_i|^{-3}$, $-|x - a_i|^{-3}$. Since $q_i \neq 0$, the map $D_{a_i} F$ is invertible. In particular, $D_a F$, and hence DF , is surjective everywhere. Thus F is a submersion and $\mathcal{Z} = F^{-1}(0)$ is a smooth submanifold of $\mathcal{E}$ of dimension $\dim \mathcal{E} - 3 = 3m$.

Let $\pi : \mathcal{Z} \rightarrow \mathcal{A}$ be the projection. At $(a, x) \in \mathcal{Z}$, a tangent vector $(\dot{a}, \dot{x})$ satisfies

$$D_a F(a, x) \dot{a} + D_x F(a, x) \dot{x} = 0.$$

It follows that $D\pi_{(a,x)}$ is surjective exactly when $D_x F(a, x)$ is invertible. Indeed, invertibility of $D_x F$ permits one to solve the displayed equation for $\dot{x}$ for every $\dot{a}$. Conversely, if $D\pi_{(a,x)}$ is surjective, then for any $y \in \mathbb{R}^3$, surjectivity of $D_a F$ gives a $\dot{a}$ with $D_a F \dot{a} = y$, and surjectivity of $D\pi$ gives a $\dot{x}$ satisfying the displayed equation. Hence $y = -D_x F \dot{x}$, so $D_x F$ is surjective and therefore invertible.

Writing $V_{a,q}$ for the potential with site vector a and charge vector q , we have $\pi^{-1}(a) = \{a\} \times \operatorname{Crit}(V_{a,q})$, so this says that a is a regular value of π exactly when every equilibrium for that site configuration is nondegenerate, because $D_x F = D_x^2 V_{a,q}$. The map π is smooth between manifolds of the same dimension $3m$, so Sard's theorem [10, Ch. 3, §1, Theorem 1.3] implies that its critical values have measure zero. No compactness or properness of π is required here: $\mathcal{Z}$ is a second-countable smooth submanifold of a Euclidean open set, so Sard's theorem applies using countably many coordinate charts. Consequently, its regular values are dense in the open set $\mathcal{A} \subset \mathbb{R}^{3m}$.

For site configurations in a sufficiently small neighborhood $\mathcal{U} \subset \mathcal{A}$ of a^0 , all sites remain a positive distance from the sets $\overline{U}_j$, and the potential depends continuously on the sites in C^2 on their union. Shrinking $\mathcal{U}$ if necessary, the K marked equilibria therefore persist for every $a \in \mathcal{U}$. Choose a regular value $a \in \mathcal{U}$ of π . Then all equilibria are nondegenerate, the K marked ones remain, and the fixed charge vector still has $Q \neq 0$. Lemma 5.1 now shows that the entire critical set is finite. $\square$

Proof of Theorem 1.1. For fixed L , Lemmas 4.1 and 4.2 produce a two-ray Coulomb sum with $2L - 1$ sites and at least B_L marked nondegenerate critical points. For every $n \geq 2L - 1$, apply Lemma 5.2 with $m = n$. This gives exactly n distinct sites, nonzero strengths, nonzero total charge, and a finite Morse critical set with at least B_L points. The original strengths include both signs by Lemma 4.2, and the sign-preserving clause of Lemma 5.2 retains them. Thus the configuration satisfies every assertion in the first part of the theorem and is admissible in the definition of $M(n)$.

The bounds on g_k are those of Lemma 3.3. In particular, absorbing the finitely many initial terms into the error term gives

$$S_L = \sum_{k=1}^{L-1} g_k = \frac{1}{2} \sum_{k=1}^{L-1} k^2 + O\left(\sum_{k=1}^{L-1} k\right) = \frac{1}{6}L^3 + O(L^2).$$

Hence $B_L = S_L + L - 1 = \frac{1}{6}L^3 + O(L^2)$ as well. Given $n \geq 3$, choose $L = \lceil n/2 \rceil$. Then $2L - 1 \leq n$, so the first part and Lemma 5.2 give

$$M(n) \geq B_{\lceil n/2 \rceil} = S_{\lceil n/2 \rceil} + \lceil n/2 \rceil - 1.$$

Since $L = n/2 + O(1)$, one has $B_{\lceil n/2 \rceil} = n^3/48 + O(n^2)$. Thus, for some fixed constant $C > 0$ and every sufficiently large integer n ,

$$M(n) \geq \frac{n^3}{48} - Cn^2 = \left(\frac{1}{48} - \frac{C}{n}\right)n^3,$$

which gives the asserted asymptotic lower bound along all integers.

For the explicit thresholds, we use the exact finite counts from Lemma 3.4 for the base cases and the analytic lower bound thereafter. Note first that $B_{L+1} - B_L = g_L + 1$.

Even n . Write $n = 2L$. By Lemma 3.4, $B_{22} = 1840 + 21 = 1861 > 1849 = 43^2$, so $M(44) > 43^2$. Passing from L to $L + 1$ raises the benchmark by

$$(2L + 1)^2 - (2L - 1)^2 = 8L,$$

so $B_L > (2L - 1)^2$ propagates to $B_{L+1} > (2L + 1)^2$ as soon as $g_L + 1 > 8L$. The exact table gives

$$(g_{22}, g_{23}, g_{24}, g_{25}, g_{26}) = (262, 286, 308, 336, 362),$$

each exceeding the corresponding values (176, 184, 192, 200, 208) of $8L$; for $L \geq 27$, Lemma 3.3 gives $g_L \geq \frac{1}{2}L^2 - 5L - 6 > 8L$, the last inequality being $L^2 - 26L - 12 > 0$; its left-hand side equals 15 at $L = 27$ and increases thereafter, since its increment from L to $L + 1$ is $2L - 25 > 0$. By induction $B_L > (2L - 1)^2$ for every $L \geq 22$, that is, $M(n) > (n - 1)^2$ for every even $n \geq 44$.

Odd n . Write $n = 2L - 1$, so the benchmark is $(2L - 2)^2$. By Lemma 3.4, $B_{21} = 1600 + 20 = 1620 > 1600 = 40^2$, so $M(41) > 40^2$. Passing from L to $L + 1$ raises the benchmark by

$$(2L)^2 - (2L - 2)^2 = 8L - 4,$$

so the induction propagates as soon as $g_L + 1 > 8L - 4$. The exact table gives

$$(g_{21}, g_{22}, g_{23}, g_{24}, g_{25}, g_{26}) = (240, 262, 286, 308, 336, 362),$$

which exceed the corresponding values $(164, 172, 180, 188, 196, 204)$ of $8L - 4$. For $L \geq 27$, Lemma 3.3 gives $g_L \geq \frac{1}{2}L^2 - 5L - 6 > 8L - 4$, the last inequality being $L^2 - 26L - 4 > 0$. Its left-hand side equals 23 at $L = 27$ and increases thereafter by the same increment $2L - 25 > 0$. Thus $M(2L - 1) \geq B_L > (2L - 2)^2$ for every $L \geq 21$, that is, $M(n) > (n - 1)^2$ for every odd $n \geq 41$.

The two cases together give $M(41) > 40^2$ and $M(n) > (n - 1)^2$ for every $n \geq 43$, which is the final assertion. For comparison, at $n = 42$ the general bound gives $M(42) \geq B_{21} = 1620$, whereas $41^2 = 1681$. Thus this estimate does not settle $n = 42$; it is not an upper bound on $M(42)$ or an assertion that the constructed configuration has exactly 1620 equilibria. $\square$

6 Why the two-degree construction uses two axes

The off-axis mechanism in this paper uses two consecutive homogeneous degrees with different zonal axes. The following proposition explains why the same two-degree mechanism cannot work with a single axis.

Proposition 6.1 (Coaxial consecutive degrees). *Let $w \in \mathbb{R}^3$ be a unit vector, let $k \geq 1$ be an integer, and let $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. The polynomial*

$$H = aZ_k^w + bZ_{k+1}^w$$

has no critical point outside the axis $\mathbb{R}w$.

Proof. Every point outside $\mathbb{R}w$ has spherical coordinates (r, θ, ϕ) about w with $r > 0$ and $0 < \theta < \pi$. Put $\mu = \cos \theta \in (-1, 1)$. Each Z_l^w is independent of ϕ , so its gradient has zero e_ϕ -component. In the orthonormal spherical frame (e_r, e_θ, e_ϕ) ,

$$\nabla Z_l^w = r^{l-1} (lP_l(\mu)e_r - \sin \theta P'_l(\mu)e_\theta).$$

The determinant of the (e_r, e_θ) component vectors of ∇Z_k^w and ∇Z_{k+1}^w is

$$\begin{aligned} \det \begin{pmatrix} kr^{k-1}P_k & (k+1)r^kP_{k+1} \\ -r^{k-1}\sin \theta P'_k & -r^k\sin \theta P'_{k+1} \end{pmatrix} \\ = r^{2k-1} \sin \theta G_k(\mu), \end{aligned}$$

where

$$G_k = (k+1)P_{k+1}P'_k - kP_kP'_{k+1}.$$

Here and below, every Legendre polynomial in this calculation is evaluated at μ . The identities $(1 - \mu^2)P'_l(\mu) = l(P_{l-1}(\mu) - \mu P_l(\mu))$, applied with $l = k$ and $l = k + 1$, give

$$(1 - \mu^2)G_k(\mu) = -k(k+1)(P_k(\mu)^2 - P_{k-1}(\mu)P_{k+1}(\mu)).$$

For completeness, the strict Legendre Turán inequality [19] follows here directly from the three-term recurrence. Write $\Delta_j = P_j^2 - P_{j-1}P_{j+1}$ for $j \geq 1$. One has $\Delta_1 = (1 - \mu^2)/2 > 0$. Applying the recurrence $(j+1)P_{j+1} = (2j+1)\mu P_j - jP_{j-1}$ at two adjacent degrees gives, for $j \geq 2$,

$$(j+1)\Delta_j = (j-1)\Delta_{j-1} + (P_j - \mu P_{j-1})^2 + (1 - \mu^2)P_{j-1}^2.$$

Induction therefore gives $\Delta_j > 0$ for every $j \geq 1$ and $-1 < \mu < 1$. Since $1 - \mu^2 > 0$, it follows that $G_k(\mu) < 0$. Moreover, $r^{2k-1} \sin \theta > 0$, so the displayed determinant is nonzero. Thus the two gradients are linearly independent at every point outside the axis. Because $(a, b) \neq (0, 0)$, their linear combination ∇H cannot vanish there. $\square$

Thus a chain of coaxial zonal blocks has no off-axis nondegenerate points to contribute through its two-term models in Lemma 4.1. This does not assert that a sum of more than two coaxial blocks has no off-axis critical points. Such a sum F is invariant under every rotation R_t about $\mathbb{R}w$, so any off-axis critical point x lies on a circle of critical points. Differentiating $\nabla F(R_t x) = R_t \nabla F(x) = 0$ at $t = 0$ gives

$$D^2 F(x)(w \times x) = 0.$$

Since $w \times x \neq 0$ off the axis, these critical points are degenerate.

There is also a low-degree obstruction under finite rotational symmetry. Let $N \geq 2$ be an integer, and suppose that a harmonic homogeneous polynomial Q_l of degree $l \geq 0$ is invariant under rotation through $2\pi/N$ about $\mathbb{R}w$. On the unit sphere, its azimuthal expansion has orders m with $-l \leq m \leq l$. Rotation multiplies the order- m mode by $e^{2\pi i m/N}$, so uniqueness of the Fourier coefficients permits only orders satisfying $N \mid m$. If $l < N$, only $m = 0$ can occur; the restriction of Q_l to the sphere is therefore a scalar multiple of $P_l(\cos \theta)$, and Q_l is a scalar multiple of Z_l^w .

Rotation preserves homogeneous degree. Hence, if a chain P_T as in Lemma 4.1 has this N -fold symmetry, uniqueness of its homogeneous decomposition implies that each block ζ_l has the same symmetry. Every consecutive-degree model $\zeta_k + \zeta_{k+1}$ with $1 \leq k < L$ and $k+1 < N$ is consequently coaxial. If its two zonal coefficients are not both zero, Proposition 6.1 rules out off-axis critical points; if both are zero, the model is identically zero and has no nondegenerate critical points. This conclusion concerns these two-term models, not all critical points of the full chain. For $l \geq N$, nonzero orders such as $m = \pm N$ may occur, so higher-degree invariant blocks need not be zonal.

7 Reproducibility note

Our construction and the cubic asymptotic lower bound are proved analytically. Computer assistance enters only through the exact finite counts in Lemma 3.4. These counts are used for the explicit conclusions $M(41) > 40^2$ and $M(n) > (n-1)^2$ for every $n \geq 43$, as well as for the comparison at $n = 42$. The additional $L-1$ axial points are proved analytically in Lemma 3.2.

All required certificate and source-code files, with their directory layout, are publicly available at github.com/shivakintali/Maxwell-Cubic:

```
verify_counts.py           requirements-cubic.txt
check_count_certificate.py count-certificate.json
```

On a POSIX system with Python 3.9 or newer and the `venv` module, run the following commands from the directory containing these files. They create an isolated environment, install SymPy 1.14.0 [21] and its dependency, regenerate and independently check the certificate, and compare it with the supplied one. The `&&` operators stop the sequence if a command fails:

```
python3 -m venv .venv-cubic &&
.venv-cubic/bin/python -m pip install -r requirements-cubic.txt &&
```

```
.venv-cubic/bin/python verify_counts.py \
--certificate count-certificate.regenerated.json &&
.venv-cubic/bin/python check_count_certificate.py \
count-certificate.regenerated.json &&
cmp count-certificate.json count-certificate.regenerated.json
```

Successful verification prints the counts and a checking summary; the final `cmp` command should produce no output and return exit status zero. The explicit output filename preserves the supplied certificate.

The generator `verify_counts.py` does not read that certificate or use it to determine any count. For each $2 \leq k \leq 26$, it constructs P'_k and P'_{k+1} over $\mathbb{Q}$, checks their degrees and square-freeness, computes the nonzero iterated boundary resultant R_k from Lemma 3.4, and isolates the roots of each derivative polynomial in rational intervals. It classifies every root pair by rational bounds on $\alpha^2 + \beta^2$, refining the intervals if necessary. An unresolved pair is never assigned a sign: the program raises an error if its refinement limit is reached. For the stated finite range, every pair is decided on the initial pass with interval widths at most 10^{-4} . Only after computing the counts does the generator compare them with the displayed values of $g_2, \dots, g_{26}$ and check the partial sums $S_{21} = 1600$, $S_{22} = 1840$, $S_{23} = 2102$, and $S_{24} = 2388$ before writing the JSON audit record. This record includes the SymPy version, isolating intervals, resultants, and every pair decision.

The checker `check_count_certificate.py` uses only Python's standard library and treats the certificate as untrusted data. It reconstructs the Legendre polynomials from their three-term recurrence and checks the intervals using Sturm sequences and exact integer and rational arithmetic. For each derivative polynomial separately, it verifies that the intervals are pairwise disjoint and contained in $(-1, 1)$, that each contains exactly one simple root, and that their number equals the degree. A singleton interval is checked by exact evaluation of the polynomial and its derivative. These checks establish that no root is omitted or counted twice. The checker also recomputes all R_k , verifies every strict inequality $\alpha^2 + \beta^2 < 1$ or $\alpha^2 + \beta^2 > 1$, and recomputes the counts and partial sums. It does not import SymPy or the generator.

This distinction between generation and checking matters: SymPy's root-isolation strategy may use floating-point logarithms internally, whereas every mathematical acceptance check in the independent checker uses integer or rational arithmetic. No count is accepted by rounding a root approximation or by a numerical tolerance. The certificate is reproducible, independently checkable data, not an additional hypothesis. This is a computer-assisted verification, not a formal proof-assistant certification; it relies on the correctness of the checking algorithms and their execution.

AI Usage: The author prepared an initial draft with the perpendicular-ray construction, the two-degree local model, and the quadratic production mechanism at each rung. The author then used multiple AI tools (Claude, ChatGPT and two open-weight models with open-source harnesses) and prompted these AI tools to work out the details of the equations and identify missing gaps. The author fixed those missing gaps and iteratively verified and expanded the proofs using AI tools. The author is solely responsible for the correctness of the proofs.

References

- [1] A. Abanov, N. Hayford, D. Khavinson, and R. Teodorescu, *Around a theorem of F. Dyson and A. Lenard: energy equilibria for point charge distributions in classical electrostatics*, Exposition. Math. **39** (2021), 182–196, [doi:10.1016/j.exmath.2021.03.003](https://doi.org/10.1016/j.exmath.2021.03.003).
- [2] P. Arathoon, G. Ball, and M. D. Kvalheim, *The Maxwell conjecture is false*, preprint (2026), [arXiv:2607.27197](https://arxiv.org/abs/2607.27197).
- [3] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, 2nd ed., Graduate Texts in Mathematics, vol. 137, Springer, New York, 2001. [Author-hosted text](#).
- [4] M. Bilogliadov, *Equilibria of Riesz potentials generated by point charges at the roots of unity*, Comput. Methods Funct. Theory **15** (2015), 471–491, [doi:10.1007/s40315-015-0115-1](https://doi.org/10.1007/s40315-015-0115-1).
- [5] H. Edelsbrunner, C. Fillmore, and G. Oliveira, *Counting equilibria of the electrostatic potential*, Proc. Lond. Math. Soc. (3) **132** (2026), Paper No. e70163, 32 pp., [doi:10.1112/plms.70163](https://doi.org/10.1112/plms.70163).
- [6] T. Erdélyi, J. Rosenblatt, and R. Rosenblatt, *The zero set of an electrical field from a finite number of point charges: one, two, and three dimensions*, preprint (2021), [arXiv:2106.04706v1](https://arxiv.org/abs/2106.04706v1).
- [7] A. Gabrielov, D. Novikov, and B. Shapiro, *Mystery of point charges*, Proc. Lond. Math. Soc. (3) **95** (2007), 443–472, [doi:10.1112/plms/pdm012](https://doi.org/10.1112/plms/pdm012).
- [8] A. Gabrielov, Dm. Novikov, T. Novikov, and B. Shapiro, *From 12 to 6: sharpening the three-charge bound in Maxwell’s problem*, preprint (2026), [arXiv:2607.28785](https://arxiv.org/abs/2607.28785).
- [9] A. Ghosh, P. W. Goldberg, and A. Hollender, *Computing equilibrium points of electrostatic potentials*, in *17th Innovations in Theoretical Computer Science Conference (ITCS 2026)*, LIPIcs, vol. 362, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2026, Art. 69, 22 pp., [doi:10.4230/LIPIcs.ITCS.2026.69](https://doi.org/10.4230/LIPIcs.ITCS.2026.69).
- [10] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics, vol. 33, Springer, New York, 1976.
- [11] T.-H. Kiang, *On the critical points of non-degenerate Newtonian potentials*, Amer. J. Math. **54** (1932), 92–109, [doi:10.2307/2371080](https://doi.org/10.2307/2371080).
- [12] K. Killian, *A remark on Maxwell’s conjecture for planar charges*, Complex Var. Elliptic Equ. **54** (2009), 1073–1078, [doi:10.1080/17476930903275946](https://doi.org/10.1080/17476930903275946).
- [13] S. Kintali, *A sharp Maxwell bound for three point charges*, preprint (2026), shivakintali.github.io/papers/Maxwell3.pdf.
- [14] S. Kintali, *A superlinear lower bound for Maxwell’s equilibrium problem*, preprint (2026), shivakintali.github.io/papers/Maxwell-Superlinear.pdf.
- [15] T.-L. Lee and Y.-L. Tsai, *Nine equilibrium points of four point charges on the plane*, Appl. Math. Lett. **132** (2022), Paper No. 108207, [doi:10.1016/j.aml.2022.108207](https://doi.org/10.1016/j.aml.2022.108207).

- [16] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, vol. 1, Clarendon Press, Oxford, 1873.
- [17] M. Morse and S. S. Cairns, *Critical Point Theory in Global Analysis and Differential Topology: An Introduction*, Pure and Applied Mathematics, vol. 33, Academic Press, New York, 1969.
- [18] R. Peretz, *Application of the argument principle to Maxwell's conjecture for three point charges*, Complex Var. Elliptic Equ. **58** (2013), 715–725, [doi:10.1080/17476933.2011.611939](https://doi.org/10.1080/17476933.2011.611939).
- [19] G. Szegő, *On an inequality of P. Turán concerning Legendre polynomials*, Bull. Amer. Math. Soc. **54** (1948), 401–405, [doi:10.1090/S0002-9904-1948-09017-6](https://doi.org/10.1090/S0002-9904-1948-09017-6).
- [20] G. Szegő, *Orthogonal Polynomials*, 4th ed., American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, Providence, RI, 1975.
- [21] A. Meurer, C. P. Smith, M. Paprocki, O. Čertík, S. B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J. K. Moore, S. Singh, T. Rathnayake, S. Vig, B. E. Granger, R. P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M. J. Curry, A. R. Terrel, Š. Roučka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, and A. Scopatz, *SymPy: symbolic computing in Python*, PeerJ Computer Science **3** (2017), e103, [doi:10.7717/peerj-cs.103](https://doi.org/10.7717/peerj-cs.103).
- [22] Y.-L. Tsai, *Special cases of three point charges*, Nonlinearity **24** (2011), 3299–3321, [doi:10.1088/0951-7715/24/12/002](https://doi.org/10.1088/0951-7715/24/12/002).
- [23] Y.-L. Tsai, *Maxwell's conjecture on three point charges with equal magnitudes*, Phys. D **309** (2015), 86–98, [doi:10.1016/j.physd.2015.07.007](https://doi.org/10.1016/j.physd.2015.07.007).
- [24] A. Yu. Uteshev and M. V. Yashina, *On Maxwell's conjecture for Coulomb potential generated by point charges*, in *Transactions on Computational Science XXVII*, Lecture Notes in Computer Science, vol. 9570, Springer, Berlin, 2016, 68–80, [doi:10.1007/978-3-662-50412-3_5](https://doi.org/10.1007/978-3-662-50412-3_5).
- [25] V. Zolotov, *Upper bounds for the number of isolated critical points via the Thom–Milnor theorem*, Anal. Math. Phys. **13** (2023), Paper No. 81, [doi:10.1007/s13324-023-00842-6](https://doi.org/10.1007/s13324-023-00842-6).