Lower Bounds for the Maxwell Equilibrium Function for Four, Six, Eight, Nine, and Ten Charges

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Abstract

Let $M_+(n)$ denote the supremum of the cardinalities of the critical sets of Coulomb potentials generated by n distinct positive point charges in $\mathbb{R}^3$, restricted to configurations with finite critical set. We prove

$$M_+(4) \geq 11, \qquad M_+(6) \geq 31, \qquad M_+(8) \geq 51, \qquad M_+(9) \geq 68, \qquad M_+(10) \geq 91.$$

In each case the exhibited configuration may be chosen Morse. Each lower bound is strictly larger than Maxwell's proposed value $(n-1)^2$. Our proofs exhibit three complementary mechanisms. The four-charge construction is validated by parameter-uniform exact-rational interval certificates. The six- and eight-charge constructions use explicit harmonic expansions, separated local scales, and analytic critical-point classifications. The nine- and ten-charge constructions use positive Kelvin designs to realize harmonic quintic and sextic limits; their large critical-point families are verified by exact-rational Krawczyk certificates. Transversality, compactness, and a global index identity complete the passage from the selected critical points to finite positive Morse configurations.

1 Introduction

1.1 The problem and our results

For pairwise distinct sites $a_1, \ldots, a_n \in \mathbb{R}^3$ and nonzero real strengths $q_1, \ldots, q_n$, consider the Coulomb potential

$$V_{a,q}(x) = \sum_{i=1}^{n} \frac{q_i}{|x - a_i|}, \qquad x \in \mathbb{R}^3 \setminus \{a_1, \ldots, a_n\}. \tag{1}$$

Its critical points are the equilibria of the electric field $-\nabla V_{a,q}$. A critical point is *nondegenerate* when its Hessian is invertible, and its Morse index is the number of negative Hessian eigenvalues, counted with multiplicity. A configuration is *Morse* when all its critical points are nondegenerate.

For $n \geq 1$, define

$$\operatorname{Crit}(V_{a,q}) = \{x \in \mathbb{R}^3 \setminus \{a_1, \ldots, a_n\} : \nabla V_{a,q}(x) = 0\}$$

and

$$M_+(n) = \sup \left\{ \# \operatorname{Crit}(V_{a,q}) : \begin{array}{ll} a = (a_1, \ldots, a_n) \in (\mathbb{R}^3)^n, & a_i \neq a_j \ (i \neq j), \\ q = (q_1, \ldots, q_n) \in (0, \infty)^n, & \# \operatorname{Crit}(V_{a,q}) < \infty \end{array} \right\}.$$

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The supremum is taken in the extended nonnegative integers. Define $M(n)$ in the same way, with $q \in (\mathbb{R} \setminus \{0\})^n$; thus $M(n) \geq M_+(n)$. Both definitions count distinct critical points without multiplicity and allow isolated degenerate points. The configurations constructed below satisfy the stronger Morse condition.

Two elementary facts about positive charges will be used repeatedly. At a critical point, the equation

$$\sum_{i=1}^{n} \frac{q_i(x - a_i)}{|x - a_i|^3} = 0$$

expresses x as a convex combination of the sites, with weights proportional to $q_i/|x - a_i|^3$. Hence every critical point lies in their convex hull. Moreover, near a site a_j, the term $q_j(x - a_j)/|x - a_j|^3$, whose norm is $q_j/|x - a_j|^2$, dominates the bounded contribution from the other sites. Thus a punctured neighborhood of each site contains no critical point. It follows that the critical set is contained in a compact subset of the charge-free domain. If the potential is Morse, this set is discrete and closed there, and is therefore finite.

Maxwell's original discussion used degree-weighted counts of equilibrium points and lines; the bound $(n-1)^2$ is the modern isolated-equilibrium inequality extracted from that discussion [1, 3]. To avoid any ambiguity about signs or genericity, throughout this paper the *positive-charge Morse Maxwell bound* means the assertion that every positive Morse configuration satisfies

$$\#\operatorname{Crit}(V_{a,q}) \leq (n-1)^2. \tag{2}$$

The five constructions assembled here give the following unified result.

Theorem 1.1 (Combined positive-charge counterexamples)**.** *For every $n \in \{4, 6, 8, 9, 10\}$, there is a configuration with n pairwise distinct sites in $\mathbb{R}^3$ and n strictly positive strengths whose Coulomb potential is Morse, has a finite critical set, and has at least L_n critical points, where $(L_4, L_6, L_8, L_9, L_{10}) = (11, 31, 51, 68, 91)$. In every case $L_n > (n-1)^2$.*

Proof. The asserted configurations are supplied, respectively, by Theorems 2.1, 3.1, 4.1, 5.1, and 6.1. Their lower bounds are $11, 31, 51, 68, 91$, whereas $(n-1)^2$ is, in the same order, $9, 25, 49, 64, 81$. Each of the five cited theorems includes positivity, distinctness of the sites, the Morse property, and finiteness of the full critical set. □

For completeness, we now justify the rows of Table 1 not supplied by the five main constructions. The equality $M_+(1) = 0$ is immediate because the field of one charge never vanishes. The two-charge assertion is also elementary. At a critical point, the two summands in the critical-point equation must be antiparallel, so the point lies on the open segment between the sites. If $L = |a_2 - a_1|$, $e = (a_2 - a_1)/L$, and $x = a_1 + t(a_2 - a_1)$ with $0 < t < 1$, then the two distances are tL and $(1-t)L$. Equating the force magnitudes gives $q_1/(tL)^2 = q_2/((1-t)L)^2$, whose unique solution is

$$t = \frac{\sqrt{q_1}}{\sqrt{q_1} + \sqrt{q_2}} \in (0, 1).$$

At this point, with $S = \sum_{i=1}^{2} q_i/|x - a_i|^3 > 0$, $D^2V(x) = S(3ee^{\mathsf{T}} - I)$, whose eigenvalues are $2S, -S, -S$. Thus the equilibrium is nondegenerate of index two and $M_+(2) = 1$, as also noted in [17]. For three charges, our sharp three-charge theorem [18, Theorem 1] gives an upper bound of four for arbitrary nonzero real strengths whenever the critical set is finite. Three equal positive charges at the vertices of an equilateral triangle have four nondegenerate critical points [13], so $M_+(3) = 4$. The five- and seven-charge rows follow from the positive Morse construction and its iteration in [17, Theorem 1 and Proposition 1]: for $n = 3 + 2m$, that paper obtains a finite critical set containing at least $4 + 20m$ points, all nondegenerate. The entries for $n = 4, 6, 8, 9, 10$ are proved here.

Except for the exact cases $n \le 3$, the displayed quantities are lower bounds, not assertions that the indicated cardinalities are exact. In particular, the proofs establish the stated bounds by isolating selected nondegenerate critical points and, for ten charges, also applying the global index identity. They do not generally exclude additional critical points.

Table 1 summarizes the exact small cases and the positive-charge lower bounds discussed here.

n	displayed result	Maxwell value	excess	source or principal mechanism
1	$M_+(1) = 0$	0	0	elementary
2	$M_+(2) = 1$	1	0	elementary; cf. Arathoon–Ball–Kvalheim [17]
3	$M_+(3) = 4$	4	0	sharp three-charge theorem [18]
4	$M_+(4) \ge 11$	9	2	direct interval certificate (this paper)
5	$M_+(5) \ge 24$	16	8	axial-pair construction [17]
6	$M_+(6) \ge 31$	25	6	analytic two-scale amplification (this paper)
7	$M_+(7) \ge 44$	36	8	iterated axial-pair construction [17]
8	$M_+(8) \ge 51$	49	2	five-charge seed and analytic amplification (this paper)
9	$M_+(9) \ge 68$	64	4	Kelvin design and certified quintic (this paper)
10	$M_+(10) \ge 91$	81	10	Kelvin design, certified sextic, and degree (this paper)

Table 1: Exact values for $1 \le n \le 3$ and positive-charge lower bounds for $4 \le n \le 10$, with sources as indicated. The excess is the displayed numerical value or lower bound minus $(n-1)^2$; it is not an assertion about the exact value of $M_+(n)$ when $n \ge 4$.

1.2 Motivation

The problem sits at the intersection of potential theory, real algebraic geometry, singularity theory, and Morse theory. Positivity is a substantial constraint: it confines every equilibrium to the convex hull of the sites. Moreover, the potential is harmonic away from the sites, so its Hessian has trace zero. An invertible real symmetric 3×3 matrix of trace zero is neither positive nor negative definite; hence every critical point of a Morse Coulomb potential has index one or two. For a finite positive Morse configuration, let m_j denote the number of critical points of index j. The global degree identity (proved in Proposition 3.16) fixes the signed difference $m_2 - m_1 = n - 1$, but does not bound the total $m_1 + m_2$ from above. A counterexample must therefore create many additional saddles while maintaining positivity, nondegeneracy, separation from the singular sites, and control of the full critical set. The five constructions show that this can be achieved both by direct geometric design and by engineering harmonic polynomial limits on carefully separated scales.

There is also a methodological motivation for presenting the results together. The smaller analytic constructions make the local bifurcation mechanism transparent, whereas the larger certified constructions show that positive Kelvin designs can realize substantially richer critical-point patterns. Their common transversality and degree arguments expose the shared topological framework behind otherwise different examples.

1.3 Historical and related work

Maxwell discussed equilibria of point charges in Article 113 of his 1873 treatise [1]. Morse and Cairns later posed the problem of obtaining a general upper bound [2, p. 293]. Gabrielov, Novikov, and Shapiro gave a precise modern formulation for nondegenerate equilibria and established the first general finite upper bound; for three charges, their specialized bound is twelve [3]. Tsai proved the upper bound of four for three equal positive charges [13]. In 2026, Gabrielov, Dm. Novikov, T. Novikov, and Shapiro lowered the upper bound for nondegenerate equilibria of three positive charges from twelve to six [4]. Our preprint [18] proves the sharp bound of four for arbitrary nonzero real strengths, assuming only that the critical set is finite. Lee and Tsai constructed four positive coplanar charges with nine nondegenerate equilibria [14]. Zolotov improved the general upper bounds by applying the Thom–Milnor theorem [15].

Complementary work treats several restricted or computational forms of the problem. Kiang studied critical points of nondegenerate Newtonian potentials [5]; Killian considered planar charge configurations [6]; and Peretz applied the argument principle to the three-charge problem [7]. Further analyses cover special three-charge geometries [8], roots-of-unity configurations for Riesz potentials [9], and symbolic investigations of Coulomb critical points [10]. Ghosh, Goldberg, and Hollender study the algorithmic problem of computing approximate equilibria [11], while Abanov, Hayford, Khavinson, and Teodorescu survey related equilibrium questions in classical electrostatics [12].

For a system of n charges, Edelsbrunner, Fillmore, and Oliveira proved the upper bound $2^n(3n-2)^3$ on the number of isolated equilibria after fixing the strengths and all but one site and choosing the remaining site generically. They also constructed unit-charge families with $n_\ell = 8^\ell$ whose equilibrium-to-charge ratio tends to $25/7$ [16]. More recently, Enciso and Peralta-Salas proved that every configuration of distinct same-sign charges has a finite critical set, without a nondegeneracy or genericity assumption [19, Theorem 1.1]. In particular,

$$M_+(n) \le 2^{n-4}(n-1)(9n^2+9n+10) \qquad (n \ge 1).$$

Thus the finiteness restriction in the definition of $M_+(n)$ is automatic; we retain it to keep the positive and signed definitions parallel. Our constructions use the elementary compactness argument above after establishing the Morse property and do not depend on this stronger finiteness theorem.

Arathoon, Ball, and Kvalheim produced the first positive counterexample: five charges with at least twenty-four nondegenerate critical points [17]. Their axial-pair iteration gives at least $4+20m$ critical points for $3+2m$ charges, for every integer $m \ge 0$, and in particular forty-four points for seven charges. Equivalently, it gives

$$M_+(n) \ge 10n - 26 \qquad (n \ge 3 \text{ odd}).$$

The excess of this lower bound over Maxwell's quantity is $4m(3-m)$; hence the iteration by itself gives strict violations only for five and seven charges. For even $n \ge 4$, start with the corresponding $(n-1)$-charge positive Morse configuration and adjoin a unit positive charge at a site tending to infinity. Choose disjoint neighborhoods of *every* point in the full original finite critical set. On their union, the added potential tends to zero in C^2, so all original points persist with their indices when the new site is sufficiently far away. A sufficiently small generic perturbation of the sites, with all strengths fixed, then makes the full configuration Morse while preserving distinct sites and these continuations. The site-perturbation argument in Section 5.7 applies to any number of charges. The continuations of the full original critical set have signed index difference $n-2$, whereas the global identity for the resulting n-charge configuration requires $n-1$; because every new Morse critical point contributes either -1 or $+1$ to this difference, at least one additional critical point must occur. See Proposition 4.11. Applying this to the preceding odd lower bound gives

$$M_+(n) \geq 10n - 35 \qquad (n \geq 4 \text{ even}).$$

Since $(10n - 35) - (n - 1)^2 = -(n - 6)^2$, that extension does not yield a strict violation; it reaches equality only for six charges. The additional innermost scale in our six- and eight-charge constructions provides the stronger counts needed in those cases. Using a different positive cubature construction we gave the asymptotic lower bound $M_+(n) = \Omega(n^{3/2})$ [20].

The computer-assisted portions below use interval Newton–Krawczyk methods in the form developed in standard references on interval analysis [23]. Arbitrary-precision ball arithmetic is supplied by Arb [24], and the transversality steps use standard differential-topological results in the forms recorded by Hirsch [21] and Guillemin–Pollack [22]. The ten-charge search also uses polynomial-system computations of the type implemented in `msolve` [25]; all computational acceptance decisions are subsequently rechecked by exact-rational or outward-rounded interval arithmetic. The complete directories `maxwell4/`, `maxwell9/`, and `maxwell10/`, including all certificates and verifier source code used in these arguments, are publicly available at

https://github.com/shivakintali/Maxwell-Lower-Bounds.

1.4 Overview of the proof strategies

The five bounds arise from three complementary constructions.

(i) *Direct certified geometry for four charges.* Four explicit rational sites and a positive box of strengths are chosen. Eleven disjoint rational cubes in physical space are accompanied by preconditioners. Parameter-uniform Krawczyk inclusions prove that every strength vector in the box has one nondegenerate equilibrium in each cube. A generic strength vector in the interior of the box makes the entire potential Morse without destroying the eleven selected points.

(ii) *Analytic scale separation for six and eight charges.* Here D_3 denotes the order-six symmetry group of an equilateral triangle, acting in the xy-plane. The seeds used below are invariant under this group and under reflection across $z = 0$. Their expansions have positive scalar coefficients multiplying the specific quadratic and cubic harmonics H_2 and H_3 defined in Section 3.2. Adding a small symmetric axial pair produces a quartic polynomial limit with twenty-one explicitly classified critical points. Seventeen of these points persist at outer scales after a still smaller one-sided charge is added; an eleven-point innermost limit supplies the remaining selected family. A triangle seed contributes three noncentral points, giving $3 + 17 + 11 = 31$ for six charges. The Arathoon–Ball–Kvalheim five-charge seed contributes twenty-three nonzero points, giving $23 + 17 + 11 = 51$ for eight charges.

(iii) *Kelvin designs for nine and ten charges.* The Kelvin parametrization represents positive charges away from the origin by positive weights and nonzero nodes whose harmonic moments determine the Taylor expansion there. A Kelvin design is such a node configuration with specified vanishing low-degree moments. For nine charges, a positive three-orbit D_3 design realizes a harmonic quintic as a rescaled local limit. Its fifty-two certified Morse critical points persist as physical equilibria alongside sixteen certified remote equilibria, yielding sixty-eight. For ten charges, the corresponding sextic supplies eighty-three persistent Morse critical points with selected index difference one. After Morse completion, the global ten-charge identity requires nine, forcing at least eight additional critical points and hence a total of at least ninety-one.

All five constructions use the same three ingredients to complete their local counts. Nondegenerate points persist under sufficiently small perturbations. A transversality argument chooses a nearby positive configuration for which every critical point is nondegenerate. Positivity then confines the critical set to a compact region separated from the sites, so discreteness implies finiteness. Computer certificates are used where a large or asymmetric system resists a transparent symbolic classification. For six and eight charges, the quartic polynomial and innermost point-charge models admit complete classifications by factorization, monotonicity, and Hessian-sign calculations; the subsequent persistence arguments establish lower bounds for the physical potentials.

1.5 Organization of the paper

Section 2 proves the four-charge bound by an exact-rational interval certificate. Sections 3 and 4 give the analytic six- and eight-charge multiscale constructions. Section 5 develops the nine-charge Kelvin design, its certified quintic, and its remote equilibria. Section 6 treats the ten-charge sextic and the degree argument forcing eight additional points. Each of the computer-assisted Sections 2, 5, and 6 ends with a self-contained description of its certificate format and reproduction commands.

1.6 Scope and limitations

Together with the five- and seven-charge constructions of Arathoon–Ball–Kvalheim, Theorem 1.1 shows that the positive-charge Morse Maxwell bound fails for every $4 \leq n \leq 10$; the cases $n = 5$ and $n = 7$ are supplied by their axial-pair construction.

The examples also illustrate two different meanings of explicitness. In the six- and eight-charge arguments, symmetry reduces the limiting critical-point equations to low-degree or monotone one-variable problems, so the counts and indices admit conventional symbolic proofs. In the four-charge argument, the sites and strength box are explicit rational data. In the nine- and ten-charge arguments, rigorously enclosed Kelvin designs and certified harmonic polynomials supply the input to local realization theorems. Exact-rational and outward-rounded interval certificates establish existence, separation, and nondegeneracy for the specified intermediate data. The final positive Morse configurations are obtained by persistence and generic perturbation; the required small scales and generic parameter choices are existential, rather than fully specified numerical configurations. Thus the certificates are rigorous proof objects, while the passage to the final configurations also uses analytic arguments.

None of the displayed lower bounds for $n \geq 4$ is asserted to be sharp. The global index identity constrains the difference between index-two and index-one critical points, but additional points may occur in cancelling index pairs. Determining the exact values of $M_+(n)$, improving general upper bounds, and finding a single scalable construction that combines the transparency of the analytic models with the density of the Kelvin designs remain natural directions.

2 Four positive charges: a direct interval certificate

2.1 The four-charge theorem and proof architecture

Theorem 2.1 (Four positive charges)**.** *There are four pairwise distinct sites and four strictly positive strengths for which the Coulomb potential has a finite critical set, every critical point is nondegenerate, and the critical set contains at least eleven points. Consequently* $M(4) \geq M_+(4) \geq 11 > 9 = (4-1)^2$.

The proof has two logically separate components. First, eleven selected nondegenerate equilibria are certified, uniformly for every strength vector in a closed box of positive strengths with nonempty interior. Second, a generic strength vector in the interior is chosen so that the full critical set is Morse and finite. The certificate proves local existence, uniqueness within each selected cube, determinant signs, separation, and persistence throughout the strength box. No claim is made that the eleven selected points exhaust the critical set.

2.2 Exact central data and a parameter box

Put $D = 10^{17}$ and define the following exact rational numbers:

$$s = 85271999344120686/D, \quad t = 77562960716758489/D, \quad w = 106969722198610960/D,$$
$$\alpha = 84379394488954595/D, \quad \beta = 158246239884357240/D.$$

The sites and the central strength vector $q^{(0)}$ are

$$\begin{aligned} a_1 &= (s, s, t), & q_1^{(0)} &= \alpha, \\ a_2 &= (s, -t, -s), & q_2^{(0)} &= \alpha, \\ a_3 &= (-t, s, -s), & q_3^{(0)} &= \alpha, \\ a_4 &= (-w, -w, w), & q_4^{(0)} &= \beta. \end{aligned} \tag{3}$$

With $a = (a_1, a_2, a_3, a_4)$ fixed as in (3), write $V_q := V_{a,q}$. All four central strengths are positive. The oriented affine determinant, with each a_i used as a three-entry column, is

$$\det \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \frac{17559026521196643344582203723377936315341234584307}{16000000000000000000000000000000000000000000000000} \neq 0.$$

Thus the sites are affinely independent; in particular, they are pairwise distinct and noncoplanar. The orthogonal map $R(x, y, z) = (y, -z, -x)$ has $R^3 = I$, cyclically permutes a_1, a_2, a_3, and fixes a_4. Since the first three central strengths are equal, $V_{q^{(0)}}$ is invariant under R. This symmetry explains the orbit sizes visible in Table 2, but no symmetry reduction is used in the proof certificate.

Let

$$\varepsilon_* = \frac{1219273}{562949953421312} = 2.16586393\ldots \times 10^{-9} \tag{4}$$

and define the closed strength box

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^4 : |q_i - q^{(0)} * i| \leq \varepsilon * * q_i^{(0)} \quad (1 \leq i \leq 4) \right\}. \tag{5}$$

Since $0 < \varepsilon_* < 1$, every vector in $\mathcal{Q}$ has four strictly positive components. Moreover, each defining interval has positive length, so $\mathcal{Q}$ has nonempty interior in $\mathbb{R}^4$.

2.3 The exact root certificate

For $d_i = x - a_i$ and $r_i = |d_i|$, differentiation gives

$$F_q(x) := \nabla V_q(x) = -\sum_{i=1}^{4} q_i \frac{d_i}{r_i^3}, \tag{6}$$

$$D_x F_q(x) = D^2 V_q(x) = \sum_{i=1}^{4} q_i \left(3 \frac{d_i d_i^{\mathsf{T}}}{r_i^5} - \frac{I}{r_i^3} \right). \tag{7}$$

The trace of (7) is zero, as expected from harmonicity.

We use the following parameter-uniform version of the Krawczyk criterion; compare [23, Chapter 8, §8.2]. Vector and matrix intervals are Cartesian products of real intervals, and arithmetic expressions involving them are evaluated by inclusion-isotonic interval arithmetic. For a real matrix $A = (a_{ij})$, write $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ for the operator norm induced by the vector supremum norm.

Lemma 2.2 (Parameter-uniform Krawczyk criterion)**.** *Let Q be a compact parameter box, let $U \subset \mathbb{R}^m$ be open, and let $F : Q \times U \to \mathbb{R}^m$ be continuous, with $F_q(x) := F(q, x)$ continuously differentiable in x for each $q \in Q$. For $r > 0$, let $X = c + [-r, r]^m \subset U$. Suppose the interval vector $[F_Q(c)]$ contains $F_q(c)$ for every $q \in Q$, and the interval matrix $[J]$ contains $D_x F_q(x)$ for every $(q, x) \in Q \times X$. For a fixed matrix $Y \in \mathbb{R}^{m\times m}$, set*

$$K = c - Y[F_Q(c)] + (I - Y[J])[-r, r]^m.$$

If $K \subset \operatorname{int} X$ and

$$\sup_{q\in Q,\, x\in X} \|I - Y D_x F_q(x)\|_\infty < 1, \tag{8}$$

then, for every $q \in Q$, F_q has exactly one zero in X. This zero lies in $\operatorname{int} X$, and the Jacobian $D_x F_q$ is invertible there. If, in addition, an interval containing $\det D_x F_q(x)$ for every $(q, x) \in Q \times X$ excludes zero, the interval certifies the sign of the Jacobian determinant at the zero.

Proof. Fix $q \in Q$ and define $T_q(x) = x - Y F_q(x)$. If $x = c + h \in X$, the integral mean-value formula gives

$$F_q(c + h) = F_q(c) + A_{q,h}h, \qquad A_{q,h} = \int_0^1 D_x F_q(c + th)\, dt.$$

Each entry of $A_{q,h}$ belongs to the corresponding entry of $[J]$. Consequently

$$T_q(c + h) = c - Y F_q(c) + (I - Y A_{q,h})h \in K,$$

so $T_q(X) \subset K \subset \operatorname{int} X$. Set

$$\theta_q = \sup_{x\in X} \|I - Y D_x F_q(x)\|_\infty < 1.$$

Because X is convex, the same integral formula applied to two points of X shows that T_q is Lipschitz on X with constant at most θ_q. Thus T_q is a contraction of the complete metric space X into itself, and the Banach fixed-point theorem gives it a unique fixed point in X.

For every $x_0 \in X$, the Neumann-series criterion shows that $Y D_x F_q(x_0) = I - (I - Y D_x F_q(x_0))$ is invertible. Because the matrices are square, both Y and $D_x F_q(x_0)$ are invertible. Consequently a fixed point of T_q is a zero of F_q, and every zero of F_q is a fixed point. The unique zero is therefore nondegenerate and, being a fixed point of T_q, belongs to $K \subset \operatorname{int} X$. Finally, its Jacobian determinant belongs to the assumed determinant interval, which proves the last assertion. □

The certificate file `maxwell4/certificate.json` contains eleven exact dyadic centers c_j and eleven exact dyadic 3×3 preconditioners Y_j. Table 2 displays decimal approximations to the centers only for readability. The displayed determinant intervals and contraction bounds refer to the central strength vector $q^{(0)}$ and the small cubes $B_j = c_j + [-2^{-22}, 2^{-22}]^3$. All witness data used in acceptance decisions and all interval endpoints are rational; square roots are enclosed between rigorously directed rational bounds. Floating-point fields in the certificate are diagnostic only.

The parameter-uniform computation uses the same centers and preconditioners, the strength box (5), and the larger cubes

$$X_j = c_j + [-2^{-17}, 2^{-17}]^3. \tag{9}$$

j	$c_{j,1}$	$c_{j,2}$	$c_{j,3}$	Morse index	κ_j	Hessian determinant enclosure
0	-1.6830723×10^{-3}	-1.6830723×10^{-3}	$+1.6830723\times10^{-3}$	2	3.167×10^{-3}	$[8.052, 8.105]\times10^{-8}$
1	$+1.6838312\times10^{-3}$	$+1.6838312\times10^{-3}$	-1.6838312×10^{-3}	1	3.179×10^{-3}	$[-8.083, -8.031]\times10^{-8}$
2	$+3.4279112\times10^{-4}$	$+3.4279112\times10^{-4}$	-1.1581423×10^{-2}	2	6.933×10^{-3}	$[6.386, 6.427]\times10^{-7}$
3	$+3.4279112\times10^{-4}$	$+1.1581423\times10^{-2}$	-3.4279112×10^{-4}	2	6.933×10^{-3}	$[6.386, 6.427]\times10^{-7}$
4	$+1.1581423\times10^{-2}$	$+3.4279112\times10^{-4}$	-3.4279112×10^{-4}	2	6.933×10^{-3}	$[6.386, 6.427]\times10^{-7}$
5	-1.6218023×10^{-2}	-2.9782060×10^{-4}	$+2.9782060\times10^{-4}$	1	1.422×10^{-2}	$[-5.960, -5.887]\times10^{-7}$
6	-2.9782060×10^{-4}	-1.6218023×10^{-2}	$+2.9782060\times10^{-4}$	1	1.422×10^{-2}	$[-5.960, -5.887]\times10^{-7}$
7	-2.9782060×10^{-4}	-2.9782060×10^{-4}	$+1.6218023\times10^{-2}$	1	1.422×10^{-2}	$[-5.960, -5.887]\times10^{-7}$
8	-4.1181093×10^{-2}	-3.5387003×10^{-4}	$+3.5387003\times10^{-4}$	2	7.412×10^{-3}	$[7.199, 7.245]\times10^{-6}$
9	-3.5387003×10^{-4}	-4.1181093×10^{-2}	$+3.5387003\times10^{-4}$	2	7.412×10^{-3}	$[7.199, 7.245]\times10^{-6}$
10	-3.5387003×10^{-4}	-3.5387003×10^{-4}	$+4.1181093\times10^{-2}$	2	7.412×10^{-3}	$[7.199, 7.245]\times10^{-6}$

Table 2: The eleven certified cubes at the central strength vector $q^{(0)}$. Here κ_j is an upper bound on the maximum absolute row sum of the interval matrix $I - Y_j[D^2V_{q^{(0)}}(B_j)]$, rounded upward. The last column encloses $\det D^2V_{q^{(0)}}(x)$ for every $x \in B_j$; each displayed interval is rounded outward. These data concern the smaller cubes B_j, not the parameter-uniform cubes X_j below.

For all eleven cubes the computation proves the strict inclusion required in Lemma 2.2. Exact interval row-sum evaluation also bounds $\sup_{q\in\mathcal{Q},\, x\in X_j} \|I - Y_j D_x F_q(x)\| * \infty$ for every j; the largest of these eleven bounds is less than 0.455. Thus condition (8) holds on every uniform cube. Every interval Hessian determinant has the same sign as in Table 2 and stays at distance more than 7.24×10^{-8} from zero. Every uniform cube is disjoint from the four sites; a direct estimate, certified with directed rational bounds for the square roots, is

$$\min *_{0\le j\le 10}^{1\le i\le 4}(|c_j - a_i| - \sqrt{3}\,2^{-17}) > 1.4123.$$

The minimum sup-norm separation of the centers is greater than 0.0033669, while the sup-norm cube diameter is $2^{-16} < 0.000015259$. Hence the eleven cubes are pairwise disjoint. All strict inequalities in this paragraph follow from exact rational interval endpoints; the displayed decimals are outward-rounded summaries only.

Proposition 2.3 (Certified selected equilibria)**.** *For every $q \in \mathcal{Q}$, the potential V_q has at least eleven distinct nondegenerate critical points, one in each cube (9). If m_k^{sel} denotes the number of these eleven points of Morse index k, then their Morse-index ledger is*

$$(m_1^{\mathrm{sel}}, m_2^{\mathrm{sel}}) = (4, 7). \tag{10}$$

Proof. Apply Lemma 2.2 to each of the eleven certified inclusions. Disjointness makes the resulting roots distinct. Since V_q is harmonic away from its sites, the three eigenvalues of a nonsingular Hessian sum to zero; hence its Morse index is either one or two. The determinant has sign $(-1)^k$ at a critical point of Morse index k. A positive determinant therefore gives index two, whereas a negative determinant gives index one. The certified interval signs yield (10). □

Two separately implemented verifiers in the supplied package replay Proposition 2.3. The primary verifier, `maxwell4/verify_certificate.py`, rounds each interval operation outward to a 2^{-240} grid. It also checks positivity and affine independence of the input configuration, separation of the cubes from the sites, pairwise cube disjointness, and explicit confinement and site-exclusion data that corroborate Section 2.4. The independent verifier, `maxwell4/independent_verify.py`, does not import the primary verifier; it keeps exact rational endpoints and encloses square roots on a 10^{-100} grid. Both implementations reconstruct equations (6) and (7), the Krawczyk sets, determinant signs, pairwise cube separation, and the quantifier over the strength box directly from `maxwell4/certificate.json`. The mutation tests in `maxwell4/test_rejections.py` are software sanity checks rather than logical ingredients of the proof.

2.4 Morse completion and finiteness

The exact central configuration already has eleven certified nondegenerate critical points. To obtain the stronger assertion in Theorem 2.1 that *every* critical point is nondegenerate, we choose a generic strength vector in the interior of the uniform parameter box. We then give an elementary proof of finiteness, independent of the general same-sign finiteness theorem cited in Section 1. No classification of the remaining critical points is needed.

Let $\mathcal{P} = \operatorname{int} \mathcal{Q}$ and $\Omega = \mathbb{R}^3 \setminus \{a_1, a_2, a_3, a_4\}$. Consider the universal gradient map

$$G : \mathcal{P} \times \Omega \longrightarrow \mathbb{R}^3, \qquad G(q, x) = \nabla V_q(x). \tag{11}$$

Lemma 2.4 (Strength submersion)**.** *For the noncoplanar sites* (3)*, the map G in* (11) *is a submersion.*

Proof. The map G is smooth on $\mathcal{P} \times \Omega$. The four columns of its derivative with respect to the strengths are

$$\partial_{q_i} G(q, x) = -\frac{x - a_i}{|x - a_i|^3}.$$

If these vectors did not span $\mathbb{R}^3$, there would be a nonzero $v \in \mathbb{R}^3$ satisfying $v \cdot (x - a_i) = 0$ for every i. Thus $v \cdot a_i = v \cdot x$ for all four sites, placing them in the affine plane $\{y \in \mathbb{R}^3 : v \cdot y = v \cdot x\}$. This contradicts the affine independence proved in Section 2.2. Hence $D_q G$ is surjective at every point of $\mathcal{P} \times \Omega$, and therefore the full derivative of G is surjective there. □

Lemma 2.5 (A regular positive strength vector)**.** *There is a vector $q^* \in \mathcal{P}$ such that every critical point of V_{q^*} is nondegenerate. The potential V_{q^*} retains the eleven critical points of Proposition 2.3.*

Proof. By Lemma 2.4, the incidence set $Z = G^{-1}(0)$ is a smooth manifold (of dimension four). Let $\pi : Z \to \mathcal{P}$ be projection onto the strength coordinate. Sard's theorem says that the critical values of π have measure zero [21, Chapter 3, §1, Theorem 1.3]. Hence the regular values of π are dense in the nonempty open set $\mathcal{P}$; in particular, $\mathcal{P}$ contains a regular value of π.

At $(q, x) \in Z$, the tangent equation is

$$D_q G\, \dot{q} + D_x G\, \dot{x} = 0.$$

If $D_x G$ is surjective, then for every prescribed $\dot{q}$ one can solve the tangent equation for $\dot{x}$; hence $D\pi$ is surjective. Conversely, let $y \in \mathbb{R}^3$. Surjectivity of $D_q G$ gives $\dot{q}$ with $D_q G\, \dot{q} = y$. If $D\pi$ is surjective, this $\dot{q}$ lifts to a tangent vector $(\dot{q}, \dot{x}) \in T_{(q,x)} Z$, and the tangent equation gives $D_x G\, \dot{x} = -y$. Thus $D\pi$ is surjective if and only if $D_x G$ is surjective. Because $D_x G = D^2 V_q(x)$ is a 3×3 matrix, surjectivity is equivalent to invertibility.

Choose a regular value $q^* \in \mathcal{P}$. Every zero of $G(q^*, \cdot)$ lies in the fiber $\pi^{-1}(q^*)$; regularity and the preceding equivalence show that its Hessian is invertible. Proposition 2.3, which is uniform on the closed box $\mathcal{Q}$, supplies the eleven selected roots for this same q^*. □

Lemma 2.6 (Compactness for positive charges)**.** *Let $n \geq 1$. For positive strengths $q_1, \ldots, q_n$ at distinct sites $a_1, \ldots, a_n \in \mathbb{R}^3$, every equilibrium lies in the convex hull of the sites. Moreover, each site has a punctured neighborhood containing no equilibrium.*

Proof. For a single site the field never vanishes, so both conclusions are immediate. Assume henceforth that there are at least two sites. At an equilibrium, set $\omega_i = q_i/|x - a_i|^3 > 0$. The equation $\nabla V(x) = 0$ becomes

$$0 = \sum_i \omega_i(x - a_i), \qquad x = \frac{\sum_i \omega_i a_i}{\sum_i \omega_i}.$$

Thus x is a convex combination of the sites (indeed, all its displayed weights are strictly positive).

Fix i and choose $\delta_i > 0$ smaller than $\frac{1}{2}\min_{k\neq i}|a_i - a_k|$. If $|x - a_i| < \delta_i$, then

$$\left|\sum_{k\neq i} q_k \frac{x - a_k}{|x - a_k|^3}\right| \leq \sum_{k\neq i} \frac{q_k}{(|a_i - a_k| - \delta_i)^2} =: C_i < \infty.$$

Choose $0 < \rho_i < \delta_i$ so that $q_i/\rho_i^2 > C_i$. Whenever $0 < |x - a_i| < \rho_i$, the magnitude $q_i/|x - a_i|^2$ of the ith gradient term is greater than C_i. That term strictly dominates the sum of the others, so the gradient cannot vanish in the punctured ball $B(a_i, \rho_i) \setminus \{a_i\}$. □

Proposition 2.7 (Finite full critical set)**.** *The potential V_{q^*} of Lemma 2.5 has a finite critical set.*

Proof. Shrink, if necessary, the radii ρ_i supplied by Lemma 2.6 so that the open balls $B(a_i, \rho_i)$ are pairwise disjoint, and set

$$K = \operatorname{conv}\{a_1, a_2, a_3, a_4\} \setminus \bigcup_{i=1}^{4} B(a_i, \rho_i).$$

This is a compact subset of Ω, and every critical point belongs to K. Since F_{q^*} is continuous on a neighborhood of K, the set $F_{q^*}^{-1}(0) \cap K$ is closed in K and therefore compact. Every zero is nondegenerate by Lemma 2.5; the inverse function theorem therefore makes every zero isolated. If the zero set were infinite, compactness would give an accumulation point in the zero set, contradicting isolation. Hence it is finite. □

Proof of Theorem 2.1. The sites are distinct by Section 2.2, and the vector q^* is strictly positive because it lies in $\mathcal{P}$. Proposition 2.3 supplies eleven distinct critical points, Lemma 2.5 makes every critical point nondegenerate, and Proposition 2.7 proves that the full critical set is finite. Hence $M_+(4) \geq 11$, and therefore $M(4) \geq M_+(4) \geq 11 > 9 = (4-1)^2$. □

2.5 Reproduction

The proof package accompanying this manuscript is the directory `maxwell4/`. The complete directory, including its certificate and verifier source code, is publicly available at

https://github.com/shivakintali/Maxwell-Lower-Bounds.

The witness data are stored in `maxwell4/certificate.json`. From the repository root, run the complete verification suite with the first command below; the second command replays the primary certificate alone.

```
python3 maxwell4/run_all.py
python3 maxwell4/verify_certificate.py
```

The package uses only the Python standard library. It was replayed with CPython 3.14.6. The driver exits with a nonzero status if a check fails and, on success, ends with `positive four-charge certificate package:  PASS`.

Sites, strengths, radii, the positive charge-box width, centers, and preconditioners are witness inputs. They are not trusted as proof conclusions: every inclusion, determinant sign, positivity test, and separation decision is recomputed by the verifiers. Floating-point numbers printed by the scripts are diagnostic approximations, not directed bounds, and are never used for an acceptance decision. Table 2 likewise displays approximate center coordinates; its contraction bounds and determinant intervals are rounded upward and outward, respectively, as specified in the caption.

3 Six positive charges: analytic two-scale amplification

3.1 The six-charge theorem and proof architecture

Theorem 3.1 (Six positive charges)**.** *There are six pairwise distinct sites in $\mathbb{R}^3$ and six strictly positive strengths whose Coulomb potential is Morse and has a finite critical set containing at least thirty-one points. Consequently $M(6) \geq M_+(6) \geq 31 > 25 = (6-1)^2$.*

The seed consists of three unit charges at the vertices of an equilateral triangle. Its three noncentral equilibria persist at a fixed positive distance from the origin. At the scale $x = \eta^2 X$, a symmetric axial pair produces the quartic polynomial P_γ; adding a smaller one-sided charge gives the first local model K_γ. Within that model, the further scale $X = d_\gamma Y$, where $d_\gamma \asymp \gamma$, resolves the effect of this charge. Seventeen outer points continued from P_γ and eleven points supplied by the innermost model coexist with the three seed points, giving $3 + 17 + 11 = 31$. A final arbitrarily small perturbation of the positive strengths removes any additional degenerate critical points while preserving these selected points. All critical-point classifications in this section are analytic; no external numerical certificate is used.

3.2 Notation and the triangle seed

Write $e_z = (0,0,1)$. In Cartesian coordinates (x,y,z) put $\rho^2 = x^2 + y^2$, and for $\rho > 0$ let θ be the azimuthal angle, so $x = \rho\cos\theta$, $y = \rho\sin\theta$. For positive quantities depending on a small parameter, both $u \asymp v$ and $u = \Theta(v)$ mean that u/v is bounded above and below by positive constants independent of that parameter. Define

$$H_2 = \rho^2 - 2z^2, \qquad H_3 = x^3 - 3xy^2 = \rho^3\cos(3\theta), \qquad H_4 = 3\rho^4 - 24\rho^2 z^2 + 8z^4.$$

These are homogeneous of degrees $2, 3, 4$ and direct differentiation gives $\Delta H_2 = \Delta H_3 = \Delta H_4 = 0$.

The triangle vertices are

$$a_1 = (1,0,0), \qquad a_2 = \left(-\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right), \qquad a_3 = \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}, 0\right),$$

so $|a_j| = 1$ and $|a_j - a_k| = \sqrt{3}$ for $j \neq k$: an equilateral triangle of circumradius one. Let

$$W(x) \;:=\; V_\triangle(x) \;=\; \sum_{j=1}^{3} \frac{1}{|x - a_j|}$$

be the seed potential, defined on $\mathbb{R}^3 \setminus \{a_1, a_2, a_3\}$. It is invariant under the planar symmetry group D_3 of the triangle and, independently, under reflection across the plane $z = 0$.

Proposition 3.2 (Expansion of the seed)**.** *W is real analytic on the open ball $B_1(0)$ and*

$$W(x) = 3 + \tfrac{3}{4}H_2(x) + \tfrac{15}{8}H_3(x) + \tfrac{9}{64}H_4(x) + O(|x|^5). \tag{12}$$

In particular $W(0) = 3$ and, writing

$$A := \tfrac{3}{4} > 0, \qquad B := \tfrac{15}{8} > 0, \qquad \mathcal{E}(x) := W(x) - W(0) - AH_2(x) - BH_3(x),$$

there are constants $0 < r_0 < 1$ and $C_0 > 0$ such that

$$|D^\alpha \mathcal{E}(x)| \le C_0|x|^{4-|\alpha|} \qquad (|x| < r_0) \tag{13}$$

for every multi-index α with $|\alpha| \le 2$.

Proof. Since the three charge positions lie on the unit sphere, $B_1(0)$ contains none of them and W is real analytic there. For a unit vector a, Taylor expansion at the origin, valid for $|x| < 1$, gives

$$\begin{aligned}\frac{1}{|x-a|} &= 1 + a\cdot x + \tfrac{1}{2}(3(a\cdot x)^2 - |x|^2) + \tfrac{1}{2}(5(a\cdot x)^3 - 3|x|^2(a\cdot x)) \\ &\quad + \tfrac{1}{8}(35(a\cdot x)^4 - 30|x|^2(a\cdot x)^2 + 3|x|^4) + O(|x|^5).\end{aligned}$$

For $\rho > 0$ one has $a_j \cdot x = \rho\cos(\theta - 2\pi(j-1)/3)$. The elementary trigonometric identities obtained by summing over these three equally spaced angles (and then extended to $\rho = 0$ by continuity) give

$$\sum_{j=1}^{3} a_j = 0, \qquad \sum_{j=1}^{3}(a_j\cdot x)^2 = \tfrac{3}{2}\rho^2, \qquad \sum_{j=1}^{3}(a_j\cdot x)^3 = \tfrac{3}{4}H_3(x), \qquad \sum_{j=1}^{3}(a_j\cdot x)^4 = \tfrac{9}{8}\rho^4.$$

Substituting, the constant term is 3; the linear term vanishes; the quadratic term is $\frac{1}{2}\left(3\cdot\frac{3}{2}\rho^2 - 3|x|^2\right) = \frac{3}{4}(\rho^2 - 2z^2) = \frac{3}{4}H_2$; the cubic term is $\frac{1}{2}\cdot 5\cdot\frac{3}{4}H_3 = \frac{15}{8}H_3$, the term $-\frac{3}{2}|x|^2\sum(a_j\cdot x)$ vanishing because $\sum a_j = 0$; and the quartic term is

$$\tfrac{1}{8}\Big(35\cdot\tfrac{9}{8}\rho^4 - 30|x|^2\cdot\tfrac{3}{2}\rho^2 + 9|x|^4\Big) = \tfrac{9}{64}(3\rho^4 - 24\rho^2z^2 + 8z^4) = \tfrac{9}{64}H_4,$$

as one checks by expanding $|x|^2 = \rho^2 + z^2$. This proves (12).

To prove (13), note that the Taylor series of $\mathcal{E}$ begins in degree four by (12). Fix $0 < r_0 < 1$. For each $|\alpha| \le 2$, put $m = 4 - |\alpha|$. Then every derivative of $D^\alpha\mathcal{E}$ of order less than m vanishes at the origin, while its derivatives of order m are bounded on $\overline{B}_{r_0}(0)$. The multivariable Taylor theorem with remainder therefore gives $|D^\alpha\mathcal{E}(x)| \le C_\alpha|x|^m$ for $|x| < r_0$. Taking the maximum of the finitely many constants C_α proves (13) with a common C_0. $\square$

Here and below, the Morse index of a nondegenerate critical point is the number of negative eigenvalues of its Hessian, counted with multiplicity. The expansion has no linear term, so $\nabla W(0) = 0$, and the quadratic part is $A(x^2 + y^2 - 2z^2)$, so

$$D^2W(0) = \operatorname{diag}(2A, 2A, -4A) = \operatorname{diag}(\tfrac{3}{2}, \tfrac{3}{2}, -3). \tag{14}$$

Since $A > 0$, the origin is a nondegenerate critical point of W of Morse index 1.

Proposition 3.3 (Four explicit seed equilibria)**.** *In addition to the origin, W has three nondegenerate critical points of Morse index 2 lying in the plane $z = 0$ at*

$$\rho = \rho_\triangle, \qquad \cos(3\theta) = -1,$$

where $\rho_\triangle$ is the unique root in $(0,\frac{1}{2})$ of

$$\rho^5+5\rho^4+\rho^3+\rho^2-4\rho+1=0. \tag{15}$$

Moreover, $\rho_\triangle\in(\frac{1}{4},\frac{1}{3})$. Thus the three-point noncentral family has Morse-index count $(m_1,m_2)=(0,3)$; including the origin, the four displayed points have count $(1,3)$.

Proof. Reflection across $z=0$ gives $\partial_z W(x,y,0)=0$. Moreover, $\partial_z W=-\sum_j z/|x-a_j|^3$, so this derivative has the sign of $-z$ when $z\neq 0$; in particular every critical point of W lies in the plane $z=0$. We need only exhibit three noncentral points, not classify the entire critical set of W. On the mirror ray $\theta=\pi$, write the point as $(-\rho,0,0)$, with $\rho\geq 0$. Reflection across the plane $y=0$, together with reflection across $z=0$, forces the y- and z-components of the gradient to vanish there. Its distances are $1+\rho$ to a_1 and $\sqrt{\rho^2-\rho+1}$ to each of a_2,a_3, and

$$\partial_x W(-\rho,0,0)=\frac{1}{(1+\rho)^2}+\frac{2\rho-1}{(\rho^2-\rho+1)^{3/2}}.$$

A zero with $\rho>0$ requires $\rho<\frac{1}{2}$ and, after clearing denominators, is equivalent to

$$(\rho^2-\rho+1)^{3/2}=(1-2\rho)(1+\rho)^2.$$

Both sides are strictly positive on $(0,\frac{1}{2})$, so squaring neither loses nor introduces a solution there. The squared equation factors as

$$(\rho^2-\rho+1)^3-(1-2\rho)^2(1+\rho)^4=-3\rho\,(\rho^5+5\rho^4+\rho^3+\rho^2-4\rho+1)=0.$$

Let f denote the quintic in (15). Since

$$f''(\rho)=20\rho^3+60\rho^2+6\rho+2>0 \qquad (\rho>0),$$

f' is strictly increasing. Also $f'(0)=-4$, $f'(\frac{1}{2})=\frac{9}{16}$, $f(0)=1$, and $f(\frac{1}{2})=-\frac{9}{32}$. Hence f' has a unique zero $c\in(0,\frac{1}{2})$, with f strictly decreasing on $(0,c)$ and strictly increasing on $(c,\frac{1}{2})$. In particular $f(c)<f(\frac{1}{2})<0$. Since $f(0)>0>f(c)$, there is exactly one root on $(0,c)$; and since f increases from $f(c)$ to the still-negative value $f(\frac{1}{2})$, there is none on $[c,\frac{1}{2})$. Thus f has exactly one root in $(0,\frac{1}{2})$. The sharper bounds follow from $f(\frac{1}{4})=\frac{101}{1024}>0$ and $f(\frac{1}{3})=-\frac{29}{243}<0$. Rotation through $2\pi/3$ supplies the other two points, and all three satisfy $\cos(3\theta)=-1$.

It remains to verify the index without relying on numerical eigenvalues. Put $r=\rho_\triangle$, $D=r^2-r+1$, $u=1+r$, and $t=1-2r$. Reflection in the planes $y=0$ and $z=0$ makes the Hessian diagonal at $(-r,0,0)$. Using

$$D^2(|x-a|^{-1})=\frac{3(x-a)(x-a)^{\mathsf T}-|x-a|^2 I}{|x-a|^5},$$

direct substitution gives its eigenvalues in the radial, unit tangential, and vertical directions as

$$\lambda_{\rm rad}=\frac{2}{u^3}+\frac{2t^2-3}{2D^{5/2}},\qquad \lambda_{\rm tan}=-\frac{1}{u^3}+\frac{6-t^2}{2D^{5/2}},\qquad \lambda_z=-\frac{1}{u^3}-\frac{2}{D^{3/2}}.$$

The critical equation says $u^{-2}=tD^{-3/2}$, while $D=(3+t^2)/4$ and $u=(3-t)/2$. Substituting $u^{-3}=t/(uD^{3/2})$ into the first two displayed eigenvalues and simplifying gives

$$\lambda_{\rm rad}=\frac{3(2t^2+3t-3)}{2(3-t)D^{5/2}},\qquad \lambda_{\rm tan}=\frac{3(6-3t-t^2)}{2(3-t)D^{5/2}}.$$

Because $r\in(\frac{1}{4},\frac{1}{3})$, one has $t\in(\frac{1}{3},\frac{1}{2})$; therefore $2t^2+3t-3$ is increasing and $6-3t-t^2$ is decreasing on this interval, and hence $2t^2+3t-3<-1<0$ and $6-3t-t^2>17/4>0$. Thus $\lambda_{\rm rad}<0$, $\lambda_{\rm tan}>0$, and $\lambda_z<0$. The point is nondegenerate of index 2, and rotational symmetry gives the same conclusion for its two copies. $\square$

3.3 The six charges

Fix the dimensionless normalization constants $\kappa = 10, \mu = \frac{1}{2}$, let $\gamma > 0$ be a free parameter, to be fixed small in Sections 3.5–3.7, and put

$$d_\gamma := \frac{\kappa\gamma}{B} = \frac{16}{3}\gamma, \tag{16}$$

using $B = \frac{15}{8}$.

For $\eta > 0$, define three additional axial sources as follows. Lemma 3.4 gives a range in which all three strengths are positive and all six sites are distinct.

1. An equal pair at the sites $\pm\eta e_z$, each of strength

$$p_{\gamma,\eta} = A\eta^3 - \gamma\eta^5 = \tfrac{3}{4}\eta^3 - \gamma\eta^5. \tag{17}$$

2. One charge at the site δe_z, where δ and its strength $Q_{\gamma,\eta}$ are given by

$$\delta = d_\gamma \eta^2 = \tfrac{16}{3}\gamma\eta^2, \tag{18}$$

$$Q_{\gamma,\eta} = \mu\gamma d_\gamma^3 \eta^8 = \frac{2048}{27}\gamma^4\eta^8. \tag{19}$$

These choices encode the two-stage normalization used in Section 3.6. After the first change of variables $x = \eta^2 X$ and division of the potential difference by η^6, the coefficients of $H_3(X)$ and $|X - d_\gamma e_z|^{-1}$ are respectively B and $Q_{\gamma,\eta}/\eta^8$. The subsequent change $X = d_\gamma Y$ and division by γd_γ^2 turn these coefficients into

$$\frac{B d_\gamma}{\gamma} = \kappa = 10, \qquad \frac{Q_{\gamma,\eta}}{\eta^8 \gamma d_\gamma^3} = \mu = \tfrac{1}{2}.$$

The formula for $p_{\gamma,\eta}$ serves a separate purpose in the first local limit: its quadratic contribution cancels the divergent term $A\eta^{-2}H_2$ from the seed and leaves γH_2, while its quartic contribution tends to $(A/4)H_4$. Section 3.4 verifies these statements with the required C^2 remainder estimates.

Denote the resulting six-charge potential by

$$V_{\gamma,\eta}(x) = W(x) + \frac{p_{\gamma,\eta}}{|x - \eta e_z|} + \frac{p_{\gamma,\eta}}{|x + \eta e_z|} + \frac{Q_{\gamma,\eta}}{|x - \delta e_z|}. \tag{20}$$

The order of parameter selection matters. First choose $\gamma > 0$ small enough to satisfy every γ-only restriction established later; with this γ fixed, choose $\eta > 0$ sufficiently small, in particular so that Lemma 3.4 applies.

Lemma 3.4 (Admissibility)**.** *Let $\gamma > 0$ and suppose*

$$0 < \eta < \min\Big\{r_0,\ \sqrt{A/\gamma},\ 1/d_\gamma\Big\}. \tag{21}$$

Then $p_{\gamma,\eta} > 0$, $Q_{\gamma,\eta} > 0$, and $0 < \delta < \eta < r_0 < 1$. Consequently the six positions $a_1, a_2, a_3, \pm\eta e_z, \delta e_z$ are pairwise distinct and the six strengths $1, 1, 1, p_{\gamma,\eta}, p_{\gamma,\eta}, Q_{\gamma,\eta}$ are all positive.

Proof. A, r_0, d_γ, and γ are positive, so the upper bound in (21) is positive and the stated range of η is nonempty. Moreover, $p_{\gamma,\eta} = \eta^3(A - \gamma\eta^2) > 0$ exactly when $\eta^2 < A/\gamma$, and $Q_{\gamma,\eta} > 0$ because μ, γ, d_γ, and η are positive. Also $\delta = d_\gamma\eta^2 > 0$, and $\delta < \eta$ is equivalent to $\eta < 1/d_\gamma$. Thus the three axial coordinates satisfy $-\eta < 0 < \delta < \eta$ and are pairwise distinct. Finally, all three new sites lie in $B_{r_0}(0)$, whereas the three already-distinct triangle vertices lie on the unit sphere. Since $r_0 < 1$, no new site coincides with a triangle vertex. $\square$

3.4 The first local limit

For $p > 0$ and $\eta > 0$, write the potential of an axial pair of strength p at $\pm\eta e_z$ as $U_{p,\eta}(x) = p|x-\eta e_z|^{-1} + p|x+\eta e_z|^{-1}$. Taylor expansion at the origin gives

$$U_{p,\eta}(x) = \frac{2p}{\eta} - \frac{p}{\eta^3}H_2(x) + \frac{p}{4\eta^5}H_4(x) + R_{p,\eta}(x), \tag{22}$$

and there is an absolute constant $C > 0$ with

$$|D^\alpha R_{p,\eta}(x)| \le C\, p\, \eta^{-7}|x|^{6-|\alpha|} \qquad (|x| \le \eta/2,\ |\alpha| \le 2). \tag{23}$$

Indeed, set

$$G(u) := |u-e_z|^{-1} + |u+e_z|^{-1}.$$

Applying the unit-vector expansion used in Proposition 3.2 to $a = e_z$ and $a = -e_z$ and adding gives

$$G(u) = 2 - H_2(u) + \tfrac{1}{4}H_4(u) + R_6(u).$$

The function G is analytic on a neighborhood of $\overline{B}_{1/2}(0)$ and satisfies $G(-u) = G(u)$. Hence all its odd-degree Taylor terms vanish, the Taylor series of R_6 begins in degree six, and the multivariable Taylor theorem gives an absolute constant $C > 0$ such that

$$|D^\alpha R_6(u)| \le C|u|^{6-|\alpha|} \qquad (|u| \le \tfrac{1}{2},\ |\alpha| \le 2).$$

Since $U_{p,\eta}(x) = (p/\eta)G(x/\eta)$, equation (22) follows with $R_{p,\eta}(x) = (p/\eta)R_6(x/\eta)$, and the chain rule gives

$$|D^\alpha R_{p,\eta}(x)| \le \frac{p}{\eta}\eta^{-|\alpha|}C\left|\frac{x}{\eta}\right|^{6-|\alpha|} = Cp\eta^{-7}|x|^{6-|\alpha|},$$

which proves (23).

Set

$$\widetilde{V}_{\gamma,\eta}(X) = \frac{V_{\gamma,\eta}(\eta^2 X) - V_{\gamma,\eta}(0)}{\eta^6},$$

and define

$$K_\gamma(X) = P_\gamma(X) + \frac{(2048/27)\gamma^4}{|X - d_\gamma e_z|}, \qquad P_\gamma(X) = \gamma H_2(X) + BH_3(X) + \frac{A}{4}H_4(X). \tag{24}$$

Proposition 3.5 (First local limit)**.** *Fix $\gamma > 0$. For every compact $\mathcal{K} \subset \mathbb{R}^3 \setminus \{d_\gamma e_z\}$,*

$$\|\widetilde{V}_{\gamma,\eta} - (K_\gamma - K_\gamma(0))\| * C^2(\mathcal{K}) = O * \gamma, \mathcal{K}(\eta^2) \qquad \textit{as } \eta \downarrow 0.$$

Proof. For the fixed γ, every sufficiently small $\eta > 0$ satisfies the admissibility conditions of Lemma 3.4; restrict to such η. Since $\mathcal{K}$ is compact and avoids $d_\gamma e_z$, it has positive distance from that point. The sixth charge sits at $d_\gamma e_z$ after rescaling, while the other scaled charge positions are $\eta^{-2}a_j$ and $\pm\eta^{-1}e_z$, which leave every compact set as $\eta \downarrow 0$. Thus all functions below are defined on a fixed neighborhood of $\mathcal{K}$ for sufficiently small η.

Seed. By homogeneity and (13), with $\mathcal{E}^W_\eta(X) := \eta^{-6}\mathcal{E}(\eta^2 X)$,

$$\frac{W(\eta^2 X) - W(0)}{\eta^6} = A\eta^{-2}H_2(X) + BH_3(X) + \mathcal{E}^W_\eta(X), \qquad \|\mathcal{E}^W_\eta\| * C^2(\mathcal{K}) = O * \mathcal{K}(\eta^2),$$

because $|D_X^\alpha[\eta^{-6}\mathcal{E}(\eta^2 X)]| = \eta^{2|\alpha|-6}|(D^\alpha\mathcal{E})(\eta^2 X)| \le C_0\eta^2|X|^{4-|\alpha|}$ by (13), uniformly for $X \in \mathcal{K}$ once $\eta^2\mathcal{K} \subset B_{r_0}(0)$.

Axial pair. Take $p = p_{\gamma,\eta} = A\eta^3 - \gamma\eta^5$. Since $U_{p,\eta}(0) = 2p/\eta$, equation (22) and the homogeneities $H_2(\eta^2 X) = \eta^4 H_2(X)$, $H_4(\eta^2 X) = \eta^8 H_4(X)$ give

$$\begin{aligned}\frac{U_{p,\eta}(\eta^2 X) - U_{p,\eta}(0)}{\eta^6} &= -\frac{p}{\eta^5}H_2(X) + \frac{p}{4\eta^3}H_4(X) + \mathcal{E}^{\mathrm{pair}} * \eta(X)\\ &= (\gamma - A\eta^{-2})H_2(X) + \Big(\frac{A}{4} - \frac{\gamma\eta^2}{4}\Big)H_4(X) + \mathcal{E}^{\mathrm{pair}} * \eta(X).\end{aligned}$$

Here $\mathcal{E}_\eta^{\mathrm{pair}}(X) = \eta^{-6}R_{p,\eta}(\eta^2 X)$. For small η one has $|\eta^2 X| \le \eta/2$ uniformly on $\mathcal{K}$, so (23) and the chain rule give, for $|\alpha| \le 2$,

$$|D_X^\alpha \mathcal{E}_\eta^{\mathrm{pair}}(X)| \le C\eta^{2|\alpha|-6}p\eta^{-7}|\eta^2 X|^{6-|\alpha|} = C\frac{p}{\eta}|X|^{6-|\alpha|}.$$

Since admissibility gives $0 < p/\eta = A\eta^2 - \gamma\eta^4 \le A\eta^2$, it follows that $\|\mathcal{E}_\eta^{\mathrm{pair}}\|_{C^2(\mathcal{K})} = O_{\gamma,\mathcal{K}}(\eta^2)$.

The two terms $\pm A\eta^{-2}H_2$ cancel exactly, and the seed-plus-pair contribution is

$$P_\gamma(X) - \frac{\gamma\eta^2}{4}H_4(X) + \mathcal{E}_\eta^W(X) + \mathcal{E}_\eta^{\mathrm{pair}}(X),$$

whose difference from P_γ is $O_{\gamma,\mathcal{K}}(\eta^2)$ in $C^2(\mathcal{K})$.

Sixth charge. By (19) and (18) one has the exact identity

$$\frac{Q_{\gamma,\eta}}{\eta^6}\Big(\frac{1}{|\eta^2 X - \delta e_z|} - \frac{1}{|\delta e_z|}\Big) = \frac{2048}{27}\gamma^4\Big(\frac{1}{|X - d_\gamma e_z|} - \frac{1}{d_\gamma}\Big),$$

with no error term at all, because $Q_{\gamma,\eta}/\eta^6 \cdot \eta^{-2} = (2048/27)\gamma^4$ and $\delta/\eta^2 = d_\gamma$. Since $P_\gamma(0) = 0$, the right-hand side is exactly the point-charge part of $K_\gamma - K_\gamma(0)$. Combining the three contributions proves the claim. $\square$

3.5 Critical points of the polynomial part

We begin with the Hessian in cylindrical coordinates. Let f be smooth near a point with $\rho > 0$, and let (e_ρ, e_θ, e_z) be the orthonormal cylindrical frame. In this frame

$$\begin{aligned}&D^2 f(e_\rho, e_\rho) = \partial_{\rho\rho}f, \quad D^2 f(e_\theta, e_\theta) = \rho^{-2}\partial_{\theta\theta}f + \rho^{-1}\partial_\rho f, \quad D^2 f(e_z, e_z) = \partial_{zz}f,\\ &D^2 f(e_\rho, e_z) = \partial_{\rho z}f, \quad D^2 f(e_\rho, e_\theta) = \rho^{-1}\partial_{\rho\theta}f - \rho^{-2}\partial_\theta f, \quad D^2 f(e_\theta, e_z) = \rho^{-1}\partial_{\theta z}f.\end{aligned} \tag{25}$$

These formulas follow, for example, from $\nabla_{e_\theta}e_\rho = \rho^{-1}e_\theta$ and $\nabla_{e_\theta}e_\theta = -\rho^{-1}e_\rho$, with the remaining derivatives of the cylindrical frame in the e_ρ and e_z directions equal to zero. Since the frame is orthonormal, this matrix is the Cartesian Hessian in a rotated basis, so its eigenvalues, determinant and signature are those of $D^2 f$.

Lemma 3.6 (Angular decoupling)**.** *Let p be a critical point of f with $\rho > 0$ and suppose $\partial_{\rho\theta}f(p) = \partial_{\theta z}f(p) = 0$. Then (25) is block diagonal at p, with meridional block $D^2_{\rho,z}f = \begin{pmatrix}\partial_{\rho\rho}f & \partial_{\rho z}f\\ \partial_{\rho z}f & \partial_{zz}f\end{pmatrix}$ and single angular eigenvalue $\rho^{-2}\partial_{\theta\theta}f$. In particular* $\det D^2 f = \rho^{-2}\partial_{\theta\theta}f \cdot \det D^2_{\rho,z}f$.

Proof. At a critical point $\partial_\rho f = \partial_\theta f = 0$, so both correction terms in (25) vanish; the hypothesis annihilates the two remaining off-block entries. $\square$

Every function to which Lemma 3.6 is applied below has the form $u(\rho, z) + c\rho^3 \cos(3\theta)$ with $c \neq 0$. For such a function $\partial_{\theta z} f \equiv 0$ and $\partial_{\rho\theta} f = -9c\rho^2 \sin(3\theta)$, which vanishes at every critical point with $\rho > 0$ because $\partial_\theta f = -3c\rho^3 \sin(3\theta) = 0$ forces $\sin(3\theta) = 0$. So the hypothesis is automatic.

Lemma 3.7 (Saddle dichotomy)**.** *Let f be harmonic on an open subset of $\mathbb{R}^3$ and p a nondegenerate critical point. Then the Morse index of p is 1 or 2; it is 1 when $\det D^2 f(p) < 0$ and 2 when $\det D^2 f(p) > 0$.*

Proof. The Hessian is symmetric with vanishing trace and no zero eigenvalue, so its three real eigenvalues are nonzero and sum to zero; hence at least one is positive and at least one negative, and the number k of negative eigenvalues is 1 or 2. The determinant is the product of the eigenvalues, so its sign is $(-1)^k$. ☐

We shall also use the following quantitative form of the stability of a nondegenerate zero. All vector norms below are Euclidean, and all matrix norms are the induced operator norms.

Lemma 3.8 (Quantitative implicit function theorem)**.** *Let $r > 0$, let $\overline{B}(x_0, r) \subset \mathbb{R}^m$, and let G be a C^1 map from a neighborhood of $\overline{B}(x_0, r)$ to $\mathbb{R}^m$. Let L be an invertible linear map, put $\nu := \|L^{-1}\|^{-1} > 0$, and assume*

$$\|DG(x) - L\| \leq \tfrac{\nu}{2} \quad \text{for all } x \in \overline{B}(x_0, r), \qquad |G(x_0)| \leq \tfrac{\nu r}{2}.$$

Then G has exactly one zero x_* in $\overline{B}(x_0, r)$, it satisfies $|x_* - x_0| \leq 2|G(x_0)|/\nu$, and $DG(x_*)$ is invertible.

Proof. Put $\Phi(x) = x - L^{-1}G(x)$, so $D\Phi(x) = L^{-1}(L - DG(x))$ and $\|D\Phi\| \leq \frac{1}{2}$ on the ball. Since the ball is convex, the mean-value inequality shows that Φ is a $\frac{1}{2}$-contraction there. Moreover, $|\Phi(x_0) - x_0| = |L^{-1}G(x_0)| \leq |G(x_0)|/\nu \leq r/2$, so for $x \in \overline{B}(x_0, r)$,

$$|\Phi(x) - x_0| \leq |\Phi(x) - \Phi(x_0)| + |\Phi(x_0) - x_0| \leq \tfrac{1}{2}|x - x_0| + \tfrac{r}{2} \leq r.$$

Hence Φ maps the ball into itself and the Banach fixed-point theorem gives a unique fixed point x_*; because L is invertible, the fixed points of Φ are exactly the zeros of G. The contraction estimate gives $|x_* - x_0| \leq (1 - \frac{1}{2})^{-1}|\Phi(x_0) - x_0| \leq 2|G(x_0)|/\nu$. Finally, $\|L^{-1}(DG(x_*) - L)\| \leq \frac{1}{2}$, so $DG(x_*) = L\big(I + L^{-1}(DG(x_*) - L)\big)$ is invertible by the Neumann series. ☐

P_γ is a linear combination of H_2, H_3, H_4, hence is harmonic on all of $\mathbb{R}^3$.

Proposition 3.9 (Critical points of P_γ)**.** *Let $\gamma > 0$ and assume*

$$\frac{A\gamma}{B^2} < \frac{3}{80}, \tag{26}$$

which for $A = \frac{3}{4}$, $B = \frac{15}{8}$ is equivalent to $\gamma < 45/256$. Under this condition, P_γ has exactly 21 critical points, all nondegenerate: ten of Morse index 1 and eleven of Morse index 2. As $\gamma \downarrow 0$, four lie in an $O(\gamma)$-neighborhood of the origin (the origin itself, of index 1, together with three points of index 2 and of norm asymptotic to $2\gamma/(3B)$); eight have norm $\Theta(\sqrt{\gamma})$; and nine converge to distinct nonzero limits. The minimum mutual distance among the eight points of norm $\Theta(\sqrt{\gamma})$ is $\Theta(\gamma)$.

Proof. In cylindrical coordinates,

$$P_\gamma = \gamma(\rho^2 - 2z^2) + B\rho^3 \cos(3\theta) + \tfrac{A}{4}(3\rho^4 - 24\rho^2 z^2 + 8z^4),$$

with first derivatives

$$\begin{aligned}
\partial_\rho P_\gamma &= \rho(2\gamma + 3B\rho\cos(3\theta) + 3A\rho^2 - 12Az^2),\\
\partial_\theta P_\gamma &= -3B\rho^3\sin(3\theta),\\
\partial_z P_\gamma &= 4z(-\gamma - 3A\rho^2 + 2Az^2).
\end{aligned}$$

When $\rho = 0$ the transverse Cartesian derivatives vanish identically on the axis and the remaining equation is $4z(-\gamma + 2Az^2) = 0$, giving the origin and the two axial points $z = \pm\sqrt{\gamma/(2A)}$. Suppose $\rho > 0$. Then $\sin(3\theta) = 0$, so $s := \cos(3\theta) \in \{+1, -1\}$. If $z = 0$ the radial equation is $2\gamma + 3Bs\rho + 3A\rho^2 = 0$, which has positive solutions only for $s = -1$, and then reads

$$3A\rho^2 - 3B\rho + 2\gamma = 0. \tag{27}$$

If $z \neq 0$ the vertical equation gives $z^2 = \gamma/(2A) + \frac{3}{2}\rho^2$; substituting into the radial equation gives $15A\rho^2 - 3Bs\rho + 4\gamma = 0$, which has positive solutions only for $s = +1$:

$$15A\rho^2 - 3B\rho + 4\gamma = 0. \tag{28}$$

The discriminants of (27) and (28) are $\Delta_p = 9B^2 - 24A\gamma$ and $\Delta_o = 9B^2 - 240A\gamma$. Condition (26) is exactly $\Delta_o > 0$, and it implies $\Delta_p > 0$. In each quadratic, the sum and product of the roots are positive; the positive discriminant makes the roots distinct. Thus both roots of each equation are positive and simple. Denote them by

$$0 < \rho_{p,-} < \rho_{p,+}, \qquad 0 < \rho_{o,-} < \rho_{o,+},$$

for the planar and off-planar equations, respectively. Thus the complete list is: the origin; two axial points; six planar points ($z = 0$, $\cos 3\theta = -1$, each planar radius supplying three angles); and twelve off-planar points ($\cos 3\theta = 1$, each off-planar radius supplying three angles and two signs of z). All these points are distinct, and their number is $1 + 2 + 6 + 12 = 21$.

Nondegeneracy. At the origin $D^2P_\gamma(0) = \mathrm{diag}(2\gamma, 2\gamma, -4\gamma)$. At either axial point, using $z^2 = \gamma/(2A)$, the eigenvalues are $(-4\gamma, -4\gamma, 8\gamma)$. At a point with $\rho > 0$ Lemma 3.6 applies and the angular eigenvalue is $\rho^{-2}\partial_{\theta\theta}P_\gamma = -9Bs\rho \neq 0$. At a planar point ($s = -1$, $z = 0$) the mixed entry vanishes, $\partial_{\rho z}P_\gamma = \partial_\rho\big(4z(-\gamma - 3A\rho^2 + 2Az^2)\big) = -24A\rho z = 0$, so the meridional block is diagonal with entries

$$3\rho(2A\rho - B) = \rho\,\tfrac{d}{d\rho}(3A\rho^2 - 3B\rho + 2\gamma) \quad\text{and}\quad -4\gamma - 12A\rho^2,$$

the second negative and the first nonzero because the roots of (27) are simple. At an off-planar point, $\partial_{\rho\rho}P_\gamma = 3\rho(B + 2A\rho)$, $\partial_{\rho z}P_\gamma = -24A\rho z$ and $\partial_{zz}P_\gamma = 16Az^2$, so the meridional determinant is

$$\begin{aligned}
\det D^2_{\rho,z}P_\gamma &= 3\rho(B + 2A\rho)\,16Az^2 - (24A\rho z)^2\\
&= 48Az^2\rho\,(B - 10A\rho),
\end{aligned} \tag{29}$$

which is nonzero because $\frac{d}{d\rho}(15A\rho^2 - 3B\rho + 4\gamma) = -3(B - 10A\rho)$ does not vanish at a simple root. Thus all 21 points are nondegenerate.

Indices. By Lemma 3.7 the index is determined by the sign of $\det D^2P_\gamma$, which for $\rho > 0$ factors as in Lemma 3.6. At the origin the determinant is $-16\gamma^3 < 0$ (index 1); at an axial point it is $128\gamma^3 > 0$ (index 2). At a planar point the angular eigenvalue $9B\rho$ is positive and the vertical eigenvalue is negative, so the sign of the determinant is opposite to that of $\frac{d}{d\rho}(3A\rho^2 - 3B\rho + 2\gamma)$. This derivative is negative at $\rho_{p,-}$ and positive at $\rho_{p,+}$; hence the determinant is positive at the smaller radius and negative at the larger. The three points of radius $\rho_{p,-}$ therefore have index 2, and the three of radius $\rho_{p,+}$ have index 1. At an off-planar point the angular eigenvalue $-9B\rho$ is negative, so the sign of the full determinant is opposite to that of (29), and hence opposite to that of $B - 10A\rho$. Since

$$\frac{d}{d\rho}(15A\rho^2 - 3B\rho + 4\gamma) = -3(B - 10A\rho),$$

one has $B - 10A\rho_{o,-} > 0$ and $B - 10A\rho_{o,+} < 0$. Thus the determinant is negative at the smaller off-planar radius and positive at the larger: the six points of radius $\rho_{o,-}$ have index 1, and the six of radius $\rho_{o,+}$ have index 2. In total

$$m_1 = 1 + 3 + 6 = 10, \qquad m_2 = 2 + 3 + 6 = 11. \tag{30}$$

Asymptotics. The exact root formulas are

$$\rho_{p,\pm} = \frac{3B \pm \sqrt{9B^2 - 24A\gamma}}{6A}, \qquad \rho_{o,\pm} = \frac{3B \pm \sqrt{9B^2 - 240A\gamma}}{30A}.$$

Expanding at $\gamma = 0$ gives

$$\rho_{p,-} = \frac{2\gamma}{3B} + O(\gamma^2), \quad \rho_{p,+} = \frac{B}{A} - \frac{2\gamma}{3B} + O(\gamma^2), \quad \rho_{o,-} = \frac{4\gamma}{3B} + O(\gamma^2), \quad \rho_{o,+} = \frac{B}{5A} - \frac{4\gamma}{3B} + O(\gamma^2).$$

The origin and the three points of radius $\rho_{p,-}$ lie in an $O(\gamma)$-neighborhood of the origin, the nonzero ones with norm asymptotic to $2\gamma/(3B)$. The two axial points and the six off-planar points of radius $\rho_{o,-}$ have norm $\Theta(\sqrt{\gamma})$, because for the latter $z^2 = \gamma/(2A) + O(\gamma^2)$. The remaining three planar points converge to $\rho = B/A$, $z = 0$, $\cos 3\theta = -1$, and the remaining six off-planar points to $\rho = B/(5A)$, $z = \pm\sqrt{\frac{3}{2}\,\frac{B}{5A}}$, $\cos 3\theta = 1$; these nine limits are nonzero and pairwise distinct. This is the decomposition $4 + 8 + 9 = 21$.

Separation of the eight. The two axial points have height $z = \pm\sqrt{\gamma/(2A)}$. The six off-planar points have radius $\rho_{o,-} = \Theta(\gamma)$ and heights $z = \pm\sqrt{\gamma/(2A) + O(\gamma^2)}$. Thus all eight have norm $\Theta(\sqrt{\gamma})$, and the six off-planar ones hug the axis at horizontal distance only $\Theta(\gamma)$ from it. For points with the same sign of z, the vertical gap from the corresponding axial point is

$$\sqrt{\frac{\gamma}{2A} + \frac{3}{2}\rho_{o,-}^2} - \sqrt{\frac{\gamma}{2A}} = \frac{(3/2)\rho_{o,-}^2}{\sqrt{\gamma/(2A) + (3/2)\rho_{o,-}^2} + \sqrt{\gamma/(2A)}} = O(\gamma^{3/2}).$$

Since the horizontal gap is $\rho_{o,-} = \Theta(\gamma)$, their distance is $\Theta(\gamma)$. Two off-planar points with the same sign of z are at distance $\sqrt{3}\rho_{o,-} = \Theta(\gamma)$, and every pair with opposite signs of z is separated by $\Theta(\sqrt{\gamma})$. Hence the minimum mutual distance among the eight is $\Theta(\gamma)$. $\square$

3.6 Eleven points on the innermost scale

Recall $d = d_\gamma = \frac{16}{3}\gamma = \kappa\gamma/B$ and put $X = dY$. By construction $Bd/\gamma = \kappa = 10$ and $(2048/27)\gamma^4/(\gamma d^3) = \mu = \frac{1}{2}$. By homogeneity, division of (24) by γd^2 gives the exact identity

$$\frac{K_\gamma(dY)}{\gamma d^2} = H_2(Y) + 10H_3(Y) + \frac{64A\gamma}{9}H_4(Y) + \frac{1}{2|Y - e_z|}, \tag{31}$$

since $Ad^2/(4\gamma) = 64A\gamma/9$. It therefore converges in C^2 on every compact subset of $\mathbb{R}^3 \setminus \{e_z\}$, as $\gamma \downarrow 0$, to

$$F(Y) = H_2(Y) + 10H_3(Y) + \frac{1}{2|Y - e_z|}. \tag{32}$$

Note that F does not depend on A or B: the normalizations κ and μ were chosen exactly to remove them. F is harmonic on $\mathbb{R}^3 \setminus \{e_z\}$, so Lemma 3.7 applies.

Lemma 3.10 (Critical points of F)**.** *F has exactly* 11 *critical points, all nondegenerate: four of Morse index* 1 *and seven of Morse index* 2*.*

Proof. Write $\mathcal{R} = \sqrt{\rho^2 + (z-1)^2}$. In cylindrical coordinates

$$\partial_\rho F = 2\rho + 30\rho^2\cos(3\theta) - \frac{\rho}{2\mathcal{R}^3}, \qquad \partial_\theta F = -30\rho^3\sin(3\theta), \qquad \partial_z F = -4z - \frac{z-1}{2\mathcal{R}^3}.$$

Axis. If $\rho = 0$ the transverse Cartesian derivatives vanish. There is no solution with $z > 1$, since both terms of $\partial_z F$ are then negative. For $z < 1$ the axial equation reduces to $z(1-z)^2 = \frac{1}{8}$ with $0 < z < 1$, the restriction $z > 0$ following from the equation itself. The cubic factors as $8z(1-z)^2 - 1 = (2z-1)(4z^2 - 6z + 1)$, whose roots are $\frac{1}{2}$ and $(3 \pm \sqrt{5})/4$; the root with the plus sign exceeds one. Hence the admissible roots are

$$z = \tfrac{1}{2}, \qquad z = \tfrac{3-\sqrt{5}}{4}. \tag{33}$$

At an axial critical point the Hessian is diagonal: in Cartesian coordinates $F = (x^2 + y^2 - 2z^2) + 10(x^3 - 3xy^2) + \frac{1}{2}\mathcal{R}^{-1}$, the Hessian of the cubic term vanishes at $x = y = 0$, and the two remaining terms are invariant under rotation about the z-axis. Using $1/(2(1-z)^2) = 4z$ from the critical equation, the repeated transverse eigenvalue and the axial eigenvalue are

$$\lambda_\perp = 2 - \frac{1}{2(1-z)^3} = \frac{2(1-3z)}{1-z}, \qquad \lambda_z = -4 + \frac{1}{(1-z)^3} = \frac{4(3z-1)}{1-z}.$$

Neither root in (33) equals $\frac{1}{3}$, so both axial points are nondegenerate; moreover $\det D^2F = \lambda_\perp^2\lambda_z$ has the sign of $3z - 1$, and since $\frac{1}{2} > \frac{1}{3} > (3-\sqrt{5})/4$, Lemma 3.7 gives index 2 at $z = \frac{1}{2}$ and index 1 at $z = (3-\sqrt{5})/4$.

Off axis. If $\rho > 0$ the angular equation gives $s = \cos(3\theta) \in \{+1, -1\}$ and the radial equation reads $\frac{1}{2\mathcal{R}^3} = 2 + 30s\rho$, so admissibility requires $2 + 30s\rho > 0$. Substituting into the vertical equation and solving for z,

$$z = \frac{2 + 30s\rho}{6 + 30s\rho}, \qquad \text{so} \qquad 1 - z = \frac{4}{6 + 30s\rho}, \tag{34}$$

and admissibility makes the numerator positive, so $0 < z < 1$. The radial equation is then equivalent to

$$C_s(\rho) := (2 + 30s\rho)\Big(\rho^2 + \frac{16}{(6+30s\rho)^2}\Big)^{3/2} = \frac{1}{2}, \tag{35}$$

and conversely every admissible positive root of (35), together with any of the three angles having the prescribed value of s, satisfies all three critical equations.

For $s = +1$, extend C_+ continuously to $\rho = 0$ and put $h = C_+^2$. A rational form is

$$h(\rho) = \frac{4(15\rho+1)^2(225\rho^4 + 90\rho^3 + 9\rho^2 + 4)^3}{729(5\rho+1)^6},$$

and direct differentiation gives

$$h'(\rho) = \frac{8\rho(15\rho+1)(225\rho^4 + 90\rho^3 + 9\rho^2 + 4)^2\,\Xi(\rho)}{243(5\rho+1)^7}, \tag{36}$$
$$\Xi(\rho) = 22500\rho^4 + 14625\rho^3 + 3375\rho^2 + 315\rho - 191.$$

Since $\Xi'(\rho) = 90000\rho^3 + 43875\rho^2 + 6750\rho + 315 > 0$ for $\rho > 0$, while $\Xi(0) = -191$ and $\Xi \to +\infty$, the polynomial Ξ has exactly one positive root ρ_*, is negative on $(0, \rho_*)$ and positive on (ρ_*, ∞). All other factors in (36) are positive for $\rho > 0$. Since $C_+ > 0$, the identity $h' = 2C_+C'_+$ shows that C_+ is strictly decreasing on $(0, \rho_*)$ and strictly increasing on (ρ_*, ∞). Exact substitution gives

$$C_+(0) > \tfrac{1}{2}, \qquad C_+(1/10) < \tfrac{1}{2}, \qquad C_+(1/4) > \tfrac{1}{2}, \tag{37}$$

the corresponding squared values being $\frac{256}{729}$, $\frac{4750104241}{21257640000}$ and $\frac{19447015044 97}{6347497291776}$: the first and third exceed $\frac{1}{4}$ while the middle one is less than $\frac{1}{4}$, and $C_+ > 0$.

We count the solutions of $C_+ = \frac{1}{2}$ on $(0, \infty)$. Since ρ_* is the global minimizer, $C_+(\rho_*) \le C_+(1/10) < \frac{1}{2}$, and no case distinction on the position of ρ_* relative to $1/10$ is needed. On $(0, \rho_*]$ the function decreases strictly from $C_+(0) = 16/27 > \frac{1}{2}$ to $C_+(\rho_*) < \frac{1}{2}$, so it attains $\frac{1}{2}$ exactly once; on $[\rho_*, \infty)$ it increases strictly from $C_+(\rho_*) < \frac{1}{2}$ to $+\infty$, since $C_+(\rho) \sim 30\rho^4$, so it attains $\frac{1}{2}$ exactly once there as well. Thus (35) has exactly two roots for $s = +1$; they lie on opposite sides of $1/10$ by (37), one in $(0, 1/10)$ and one in $(1/10, 1/4)$, and both are simple because C'_+ vanishes only at ρ_*, where the value is strictly below $\frac{1}{2}$.

For $s = -1$ admissibility is exactly $0 < \rho < 1/15$, and $C_-(\rho) = C_+(-\rho)$, where the defining expression for C_+ is positive and smooth on $(-1/15, 0)$. On this interval the positive terms of $\Xi(-\rho)$ are bounded above by $22500/15^4 + 3375/15^2 = 139/9$, so $\Xi(-\rho) < 139/9 - 191 < 0$. In (36) evaluated at $-\rho$, the factors $-\rho$ and $\Xi(-\rho)$ are both negative and every other factor is positive. Since $h = C_+^2$ and $C_+ > 0$ on this extended interval, $C'_+(-\rho) > 0$, and hence $C'_-(\rho) = -C'_+(-\rho) < 0$. Moreover $C_-(0) = \frac{16}{27} > \frac{1}{2}$ and $\lim_{\rho\uparrow 1/15} C_-(\rho) = 0$. It follows that (35) has exactly one simple root for $s = -1$.

Each root with $s = +1$ supplies three angles and the root with $s = -1$ supplies three more, giving nine off-axis points; together with the two axial points of (33) this gives 11.

Nondegeneracy and indices off axis. Lemma 3.6 applies, with angular eigenvalue $\rho^{-2}\partial_{\theta\theta}F = -90s\rho \neq 0$. Put $a_s(\rho) = 2 + 30s\rho$, $q(\rho, z) = \frac{1}{2\mathcal{R}^3}$, and

$$G_1(\rho, z) = \frac{\partial_\rho F}{\rho} = a_s(\rho) - q(\rho, z), \qquad G_2(\rho, z) = \partial_z F = -4z - (z-1)q(\rho, z).$$

At a common zero, differentiating $\partial_\rho F = \rho G_1$ gives

$$\det D^2_{\rho,z}F = \rho \det D_{(\rho,z)}(G_1, G_2). \tag{38}$$

Set $H = G_2 - (z-1)G_1 = -4z - (z-1)a_s(\rho)$; on the admissible interval, the equation $H = 0$ is exactly (34). Write its solution as $z = z_s(\rho)$ and put $\mathcal{R}_s = \mathcal{R}(\rho, z_s(\rho))$. Along this graph

$$g_s(\rho) := G_1(\rho, z_s(\rho)) = \frac{C_s(\rho) - \frac{1}{2}}{\mathcal{R}_s^3},$$

so at a root of $C_s = \frac{1}{2}$ one has $g'_s = C'_s/\mathcal{R}_s^3$. Also $H_z = -(a_s + 4)$, and implicit differentiation of $H(\rho, z_s(\rho)) = 0$ gives

$$g'_s = \frac{\det D_{(\rho,z)}(G_1, H)}{H_z} = \frac{\det D_{(\rho,z)}(G_1, G_2)}{H_z},$$

the second equality holding at the common zero because differentiating $H = G_2 - (z-1)G_1$ changes the second Jacobian row by a multiple of the first (the additional $-G_1\,dz$ term vanishes there). Finally $G_1 = 0$ gives $\mathcal{R}^{-3} = 2a_s$. Substituting in (38) yields the transparent identity

$$\det D^2_{\rho,z}F = -2\rho\, a_s(\rho)\big(a_s(\rho) + 4\big)C'_s(\rho). \tag{39}$$

On the admissible intervals $\rho > 0$, $a_s > 0$ and $a_s + 4 > 0$; since all three roots of (35) are simple, $C'_s \neq 0$, so the meridional determinant is nonzero with sign opposite to that of C'_s. All nine off-axis points are therefore nondegenerate.

By Lemma 3.7 and the factorization $\det D^2F = \rho^{-2}\partial_{\theta\theta}F \cdot \det D^2_{\rho,z}F$: at the root with $s=+1$ in $(0,\rho_*)$ one has $C'_+ < 0$, so the meridional determinant is positive while the angular eigenvalue -90ρ is negative, the full determinant is negative and the index is 1; at the root with $s=+1$ in (ρ_*,∞) one has $C'_+ > 0$, both factors are negative and the index is 2; at the root with $s=-1$ one has $C'_- < 0$, so the meridional determinant is positive while the angular eigenvalue $+90\rho$ is positive, and the index is 2. Each root contributes three points. Adding the two axial points, one of each index,

$$m_1 = 1+3 = 4, \qquad m_2 = 1+3+3 = 7. \tag{40}$$

□

Let $Y_1,\dots,Y_{11}$ be the critical points of Lemma 3.10. For each j, set $L_j = D^2F(Y_j)$ and $\nu_j = \|L_j^{-1}\|^{-1}$. Choose sufficiently small pairwise disjoint closed balls $\overline{B}(Y_j,r_j)$ that avoid e_z and satisfy

$$\|D^2F(Y) - L_j\| \le \frac{\nu_j}{4} \qquad (Y \in \overline{B}(Y_j,r_j)).$$

By (31), the functions

$$F_\gamma(Y) := \frac{K_\gamma(dY)}{\gamma d^2}$$

converge to F in C^2 on the union of these balls. Consequently, for all sufficiently small $\gamma > 0$,

$$\|D^2F_\gamma(Y) - L_j\| \le \frac{\nu_j}{2} \quad (Y \in \overline{B}(Y_j,r_j)), \qquad |\nabla F_\gamma(Y_j)| \le \frac{\nu_j r_j}{2}.$$

Lemma 3.8, applied with $G = \nabla F_\gamma$, therefore gives a unique nondegenerate critical point $Y_j(\gamma)$ in each ball. Its displacement bound tends to zero, so $Y_j(\gamma) \to Y_j$. Moreover, every matrix on the segment from L_j to $D^2F_\gamma(Y_j(\gamma))$ is within $\nu_j/2$ of L_j, and is therefore invertible by the Neumann series. The inertia of a continuous path of invertible symmetric matrices is constant, so the Morse index is unchanged. The rescaling $X = dY$ gives $D^2_Y F_\gamma(Y) = \gamma^{-1}D^2_X K_\gamma(dY)$; since $\gamma > 0$, it preserves the Hessian signature. Therefore

$$X_j(\gamma) = dY_j(\gamma) = dY_j + o(d), \qquad j = 1,\dots,11, \tag{41}$$

are 11 distinct nondegenerate critical points of K_γ, all of norm $O(d) = O(\gamma)$ and with the Morse-index count $(m_1,m_2) = (4,7)$ of (40).

3.7 Persistence and the total count

Write $S_\gamma(X) = (2048/27)\gamma^4|X - d_\gamma e_z|^{-1}$, so that $K_\gamma = P_\gamma + S_\gamma$, and note that for $j = 0,1,2$

$$|D^jS_\gamma(X)| \le C_j\gamma^4|X - d_\gamma e_z|^{-1-j}. \tag{42}$$

Here $|D^jS_\gamma|$ denotes the operator norm of the jth derivative (with the usual absolute value when $j=0$). Call the four critical points of P_γ lying in an $O(\gamma)$-neighborhood of the origin the *inner* ones, and the remaining 17 the *outer* ones; the latter comprise the eight of norm $\Theta(\sqrt{\gamma})$ and the nine with nonzero limits. This section shows that all 17 outer points persist in K_γ, and then that the three seed points of Proposition 3.3 persist in $V_{\gamma,\eta}$.

The delicate case is the eight points of norm $\Theta(\sqrt{\gamma})$, whose Hessians degenerate at rate γ while their minimum mutual separation is only $\Theta(\gamma)$. We therefore apply Lemma 3.8 with constants uniform in these eight moving base points.

Proposition 3.11 (Persistence of the outer points). *For all sufficiently small $\gamma > 0$, K_γ has 17 distinct nondegenerate critical points, one near each outer critical point of P_γ, with the same Morse indices and hence with index count $(m_1, m_2) = (9, 8)$. Together with (41), and since the 11 inner points have norm $O(\gamma)$ while the 17 outer ones have norm $\asymp \sqrt{\gamma}$ or $\asymp 1$, the two families are disjoint and*

$$\#\,\mathrm{Crit}(K_\gamma) \;\geq\; 17 + 11 = 28. \tag{43}$$

Proof. The eight points of norm $\Theta(\sqrt{\gamma})$. At the two axial points the eigenvalues divided by γ are $(-4, -4, 8)$. At each of the six off-planar points of radius $\rho_{o,-}$ the angular eigenvalue and the meridional Hessian, again divided by γ, have the limits

$$\gamma^{-1}\lambda_\theta \longrightarrow -12, \qquad \gamma^{-1}D^2_{\rho,z}P_\gamma \longrightarrow \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix},$$

which follow from $\rho_{o,-} = 4\gamma/(3B) + O(\gamma^2)$ and $z^2 = \gamma/(2A) + O(\gamma^2)$: indeed $\lambda_\theta = -9B\rho_{o,-} = -12\gamma + O(\gamma^2)$, $\partial_{\rho\rho}P_\gamma = 3\rho(B + 2A\rho) = 4\gamma + O(\gamma^2)$, $\partial_{zz}P_\gamma = 16Az^2 = 8\gamma + O(\gamma^2)$ and $\partial_{\rho z}P_\gamma = -24A\rho z = O(\gamma^{3/2})$. These limits are independent of A and B. Since an orthonormal cylindrical frame is an orthogonal change of basis, the eight Cartesian Hessians divided by γ have all eigenvalues bounded away from zero, uniformly for small γ. Thus there are $\gamma_0 > 0$ and $\nu_0 > 0$ with $\|(\gamma^{-1}D^2P_\gamma(X_0))^{-1}\|^{-1} \geq \nu_0$ at each of the eight points X_0 whenever $0 < \gamma < \gamma_0$.

By the last paragraph of the proof of Proposition 3.9 the minimum mutual distance among these eight points is $\Theta(\gamma)$. Consequently, there is a constant $c_{\mathrm{sep}} > 0$ such that, for all sufficiently small γ, the closed balls of radius $c_{\mathrm{sep}}\gamma$ about these points are pairwise disjoint. Moreover, uniformly on these balls, D^3P_γ is bounded by a constant multiple of $B + A\sqrt{\gamma} = O(1)$, because D^3H_3 is constant and D^3H_4 is linear. Hence, for every fixed $0 < c \leq c_{\mathrm{sep}}$ and $X \in \overline{B}(X_0, c\gamma)$,

$$\|D^2P_\gamma(X) - D^2P_\gamma(X_0)\| \leq c\gamma \sup_{\overline{B}(X_0,c\gamma)} \|D^3P_\gamma\| = O(c\gamma),$$

where the implicit constant is independent of c, γ, and the choice of X_0. We may therefore fix $c \in (0, c_{\mathrm{sep}}]$ sufficiently small that $\|D^2P_\gamma(X) - D^2P_\gamma(X_0)\| \leq \nu_0\gamma/4$ throughout every such ball. On these balls $|X - d_\gamma e_z| \asymp \sqrt{\gamma}$, since $d_\gamma = \Theta(\gamma)$, so (42) gives $|\nabla S_\gamma| = O(\gamma^3)$ and $\|D^2S_\gamma\| = O(\gamma^{5/2}) = o(\gamma)$.

Apply Lemma 3.8 with $G = \nabla K_\gamma$, $m = 3$, $r = c\gamma$ and $L = D^2P_\gamma(X_0)$, so $\nu \geq \nu_0\gamma$. The first hypothesis holds because

$$\|D^2K_\gamma(X) - L\| \leq \|D^2P_\gamma(X) - D^2P_\gamma(X_0)\| + \|D^2S_\gamma(X)\| \leq \tfrac{\nu_0\gamma}{4} + O(\gamma^{5/2}) \leq \tfrac{\nu_0\gamma}{2} \leq \tfrac{\nu}{2}$$

for small γ; the second holds because $\nabla P_\gamma(X_0) = 0$ gives $|G(X_0)| = |\nabla S_\gamma(X_0)| = O(\gamma^3) \leq (\nu_0 c/2)\gamma^2 \leq \nu r/2$. The lemma therefore yields exactly one critical point of K_γ in each ball, it is nondegenerate, and its displacement from X_0 is $O(\nu^{-1}\gamma^3) = O(\gamma^2) = o(\gamma)$. As the balls are disjoint, the eight continuations are distinct. Throughout each ball the straight-line family from $D^2P_\gamma(X_0)$ to $D^2K_\gamma(X)$ remains invertible by the same norm estimate, so each continuation has the same Morse index as its parent.

The nine points with nonzero limits. At a limiting planar point the angular, radial and vertical eigenvalues are $9B^2/A$, $3B^2/A$ and $-12B^2/A$; at a limiting off-planar point (29) has the nonzero limit $-72B^4/(125A^2)$ while the angular eigenvalue tends to $-9B^2/(5A)$. Thus all nine limiting Hessians are invertible. Put $P_0 = BH_3 + (A/4)H_4$. The nine limits listed in Proposition 3.9 are therefore nondegenerate critical points of P_0. Choose fixed pairwise disjoint closed balls about them, with closures avoiding the origin. If Z is such a limiting point, put $L_Z = D^2P_0(Z)$ and

$\nu_Z = \|L_Z^{-1}\|^{-1}$. Choose the radius of its ball so small that $\|D^2 P_0(X) - L_Z\| \le \nu_Z/4$ throughout the ball. On the union of these finitely many balls, $P_\gamma \to P_0$ in C^2. Moreover, the distance to $d_\gamma e_z$ is bounded below uniformly for small γ, so (42) gives $\|S_\gamma\|_{C^2} = O(\gamma^4)$. Hence $K_\gamma \to P_0$ in C^2 there. Lemma 3.8, with $G = \nabla K_\gamma$, $x_0 = Z$, and $L = L_Z$, now gives a unique nondegenerate critical point of K_γ in each ball for all sufficiently small γ. The explicit formulas in Proposition 3.9 show that the corresponding critical point of P_γ lies in the same ball and that both points converge to Z. Their Hessians converge to the same invertible matrix L_Z, so their Morse indices agree.

This proves that all 17 outer critical points of P_γ persist in K_γ with unchanged indices. Their index count is $(9, 8)$, because the four omitted inner points of P_γ consist of the index-1 origin and the three index-2 points of radius $\rho_{p,-}$. The 11 inner points of (41) satisfy $|X| = O(d) = O(\gamma)$, whereas the 17 outer points have distance $\asymp \sqrt{\gamma}$ or $\asymp 1$ from the origin, so the two families are disjoint for small γ. □

Proposition 3.12 (Persistence on the physical scale)**.** *For every sufficiently small $\gamma > 0$ and, after γ is fixed, every sufficiently small admissible $\eta > 0$, the potential $V_{\gamma,\eta}$ has at least 31 distinct nondegenerate critical points. The selected points have Morse-index count $(m_1^{\mathrm{sel}}, m_2^{\mathrm{sel}}) = (13, 18)$.*

Proof. Fix $\gamma > 0$ small enough that (26), the eleven inner continuations in (41), and all the smallness conditions in Proposition 3.11 hold simultaneously; in particular $\gamma < 45/256$. Keep this γ fixed throughout the choice of η.

For each of the 28 selected critical points X_j of K_γ, put $L_j = D^2 K_\gamma(X_j)$ and $\nu_j = \|L_j^{-1}\|^{-1}$. Choose pairwise disjoint closed balls $\overline{B}(X_j, r_j)$ whose union avoids $d_\gamma e_z$ and on which $\|D^2 K_\gamma - L_j\| \le \nu_j/4$. Proposition 3.5 implies, simultaneously on these finitely many balls, that for all sufficiently small $\eta > 0$,

$$\|D^2 \widetilde{V}_{\gamma,\eta} - D^2 K_\gamma\| \le \frac{\nu_j}{4} \quad \text{on } \overline{B}(X_j, r_j), \qquad |\nabla \widetilde{V}_{\gamma,\eta}(X_j)| \le \frac{\nu_j r_j}{2}.$$

The triangle inequality consequently gives $\|D^2 \widetilde{V}_{\gamma,\eta} - L_j\| \le \nu_j/2$ on each ball. Thus both hypotheses of Lemma 3.8, with $G = \nabla \widetilde{V}_{\gamma,\eta}$, are satisfied. The lemma gives a unique nondegenerate critical point of $\widetilde{V}_{\gamma,\eta}$ in each ball. Its Hessian can be joined to L_j through invertible symmetric matrices, so its Morse index equals that of X_j. Moreover,

$$\nabla_X \widetilde{V}_{\gamma,\eta}(X) = \eta^{-4} \nabla_x V_{\gamma,\eta}(\eta^2 X), \qquad D_X^2 \widetilde{V}_{\gamma,\eta}(X) = \eta^{-2} D_x^2 V_{\gamma,\eta}(\eta^2 X).$$

Thus the invertible rescaling $x = \eta^2 X$ preserves critical points, nondegeneracy, and Morse index, and these continuations yield 28 distinct critical points of $V_{\gamma,\eta}$. Their X-coordinates remain in a fixed bounded set, so all 28 lie at distance $O(\eta^2)$ from the origin.

The three critical points of W away from the origin, given by Proposition 3.3, have positive distance $\rho_\triangle$ from it. Choose sufficiently small mutually disjoint closed balls about them whose union avoids the three triangle vertices and the origin. On the union of these balls, the distances to the three new axial sites are bounded below uniformly for small η. The potentials of the axial pair and of the sixth charge therefore tend to zero in C^2: their strengths are respectively $O(\eta^3)$ and $O(\eta^8)$, and all three sites tend to the origin. Shrinking the balls, if necessary, the seed Hessian varies by less than one quarter of its least singular value on each ball. The preceding C^2 convergence supplies the other quarter and makes the perturbing gradient at the center arbitrarily small. Hence Lemma 3.8, applied to $\nabla V_{\gamma,\eta}$ with the seed Hessian at the center as reference matrix, gives a unique nondegenerate critical point in each ball for sufficiently small η, with unchanged Morse index 2. These three points remain a fixed positive distance from the origin and are consequently disjoint from the 28 local points.

Finally, take any $\eta > 0$ small enough that both persistence conclusions hold and that the admissibility conditions of Lemma 3.4 are satisfied. This respects the parameter order γ then η and produces six pairwise distinct sites with strictly positive strengths for which $V_{\gamma,\eta}$ has at least

$$3 + 28 = 3 + 17 + 11 = 31 \tag{44}$$

distinct nondegenerate critical points. Their Morse-index count is

$$\underbrace{(0,3)}_{\text{seed}} + \underbrace{(9,8)}_{\text{outer}} + \underbrace{(4,7)}_{\text{inner}} = (13,18),$$

by Proposition 3.3, Proposition 3.11, and (41). $\square$

Remark 3.13 (Bookkeeping). It is tempting to describe the sixth charge as replacing the four inner critical points of P_γ by 11 points, so that $3 + 21 - 4 + 11 = 31$. This is only a mnemonic. The proof instead supplies $3 + 17 + 11 = 31$ distinct nondegenerate critical points; it neither asserts nor requires that the four inner points of P_γ disappear. A continuation of one of those four points need not be distinct from the eleven already counted inner points; only additional critical points distinct from all selected points would increase the lower bound.

Remark 3.14 (Why γ must be free and small). The explicit condition (26) guarantees that P_γ has the required 21-point pattern. The inner-limit argument and the persistence estimates require the stronger asymptotic choice $\gamma \downarrow 0$: then $d_\gamma = \frac{16}{3}\gamma$, the eleven inner points lie at distance $O(\gamma)$ from the origin, and the nearest outer points lie at distance $\Theta(\sqrt{\gamma})$. Thus the proof requires a freely adjustable γ; the explicit inequality $\gamma < 45/256$ alone does not guarantee the additional smallness conditions needed for persistence. After fixing γ to satisfy all these conditions, we choose η sufficiently small depending on that fixed value.

3.8 Removing all remaining degeneracies

The 31 counted points are already nondegenerate and therefore isolated, but the argument so far does not exclude *additional* degenerate critical points. The following proposition removes that possibility while preserving the selected points and their indices.

Proposition 3.15 (Generic Morse completion). *Fix parameters γ and η for which Proposition 3.12 supplies the 31 selected nondegenerate critical points. Keep the resulting six charge positions fixed, and let $q^{(0)} \in (0,\infty)^6$ be their strength vector. Every neighborhood of $q^{(0)}$ contains a positive strength vector q' whose potential is Morse with a finite critical set and has at least 31 critical points, including continuations of the selected points with the same Morse indices.*

Proof. Keep the six positions fixed and label them $b_1, \dots, b_6$, with b_1, b_2, b_3 the triangle vertices and $b_4 = \eta e_z$. Let $\mathcal{N}$ be a prescribed neighborhood of $q^{(0)}$ in the positive orthant $\mathcal{P} = (0,\infty)^6$. Replacing $\mathcal{N}$ by a smaller set if necessary, assume that it is open in $\mathcal{P}$. On the smooth manifold $M = \mathbb{R}^3 \setminus \{b_1, \dots, b_6\}$ consider the smooth family of electric fields

$$E(q,x) = \sum_{i=1}^{6} q_i \frac{x - b_i}{|x - b_i|^3} = -\nabla V_q(x).$$

For fixed $(q,x) \in \mathcal{P} \times M$, the derivatives with respect to the strengths are $\partial_{q_i} E(q,x) = (x - b_i)/|x - b_i|^3$. These vectors span $\mathbb{R}^3$: if a nonzero v were orthogonal to all of them, then $v \cdot (x - b_i) = 0$, and hence $v \cdot b_i = v \cdot x$, for every i. All six positions would then lie in the affine plane $\{y \in \mathbb{R}^3 : v \cdot y = v \cdot x\}$.

This is impossible: the three noncollinear points b_1, b_2, b_3 determine the plane $z = 0$, whereas $b_4 = \eta e_z \notin \{z = 0\}$. Hence $D_q E(q, x)$ is surjective at every (q, x); a fortiori the full derivative of $E : \mathcal{P} \times M \to \mathbb{R}^3$ is surjective, so E is a submersion.

Choose pairwise disjoint implicit-function-theorem uniqueness neighborhoods in M of the 31 selected critical points. Applying the theorem to the map $(q, x) \mapsto E(q, x)$ at each of these points, and then intersecting the resulting finitely many parameter neighborhoods, gives an open set $\mathcal{U} \subset \mathcal{N}$ containing $q^{(0)}$. Every $q \in \mathcal{U}$ has a unique critical point in each of the 31 spatial neighborhoods. After shrinking $\mathcal{U}$ once more, continuity of the Hessian makes all these continuations nondegenerate and preserves their Morse indices.

Since a submersion is transverse to $\{0\}$, the smooth parametric transversality theorem [22, p. 68] applies to $E : \mathcal{P} \times M \to \mathbb{R}^3$ and the embedded submanifold $\{0\} \subset \mathbb{R}^3$. It implies that, for almost every $q \in \mathcal{P}$, the partial map $E_q := E(q, \cdot)$ is transverse to $\{0\}$. Its differentiability requirement is satisfied because E is smooth (here $\dim M - \operatorname{codim}\{0\} = 0$). The exceptional set has Lebesgue measure zero and therefore cannot contain a nonempty open subset of $\mathcal{P}$; the admissible parameters are consequently dense. Choose $q' \in \mathcal{U}$ with this property. At a zero of $E_{q'}$ one has $D_x E_{q'}(x) = -D^2 V_{q'}(x)$. Thus transversality of $E_{q'}$ to $\{0\}$ is equivalent to nondegeneracy of every critical point of $V_{q'}$, while the membership $q' \in \mathcal{U}$ preserves the 31 selected points and their indices.

Finiteness. Every critical point lies in the convex hull of the charge positions: at a zero of $E_{q'}$,

$$x = \frac{\sum_i (q'_i/|x - b_i|^3) b_i}{\sum_i q'_i/|x - b_i|^3},$$

which is a convex combination because every coefficient $q'_i/|x - b_i|^3$ is positive. To exclude zeros near the singularities, choose pairwise disjoint closed balls $\overline{B}_{R_i}(b_i)$, each containing no other charge, and put

$$C_i := \sup_{x \in \overline{B}_{R_i}(b_i)} \left| \sum_{j \neq i} q'_j \frac{x - b_j}{|x - b_j|^3} \right| < \infty.$$

Choose $0 < r_i < R_i$ so small that $q'_i/r_i^2 > C_i$. If $0 < |x - b_i| < r_i$, then the ith summand of $E_{q'}(x)$ has norm $q'_i/|x - b_i|^2 > C_i$, whereas the sum of the other five has norm at most C_i. Thus $E_{q'}(x) \neq 0$ throughout $B_{r_i}(b_i) \setminus \{b_i\}$.

Let H be the convex hull of the charge positions and set $K = H \setminus \bigcup_i B_{r_i}(b_i)$. The preceding two paragraphs show that every critical point belongs to K. This set is compact and avoids all charge positions, so $E_{q'}$ is continuous on K and its zero set $Z \subset K$ is compact. Transversality makes $D_x E_{q'}$ invertible at every point of Z; the inverse function theorem therefore makes every point of Z isolated. If Z were infinite, compactness would give an accumulation point in Z, contradicting this isolation. Hence Z is finite, and $V_{q'}$ is Morse with a finite critical set of cardinality at least 31. □

Proof of Theorem 3.1. Choose γ and then η as in Proposition 3.12. Apply Proposition 3.15 to the resulting strength vector. The sites remain pairwise distinct, and the perturbation may be chosen inside the positive orthant. The resulting potential is Morse, has a finite critical set, and retains the 31 selected critical points. Hence $M(6) \geq M_+(6) \geq 31 > 25 = (6 - 1)^2$. □

3.9 The Morse-index ledger

Every construction above produces critical points together with their Morse indices. We record the ledger as a consistency check and to constrain what any further critical points could look like.

Proposition 3.16 (Index identity). *Let $n \geq 1$ and $V = \sum_{i=1}^{n} q_i |x - b_i|^{-1}$ with all $q_i > 0$ and all b_i distinct, and suppose V is Morse. Let m_1, m_2 be the numbers of critical points of Morse index 1 and 2. Then every critical point has index 1 or 2, both numbers are finite, and*

$$m_2 - m_1 = n - 1, \qquad \textit{equivalently} \qquad \#\,\mathrm{Crit}(V) = (n-1) + 2m_1. \tag{45}$$

Proof. The confinement and site-exclusion arguments in the proof of Proposition 3.15 use only positivity and distinctness of the charges: the convex-combination identity confines all critical points to the compact convex hull, and the dominant singular summand excludes a punctured neighborhood of each charge. Thus $\mathrm{Crit}(V)$ is contained in a compact subset of the charge-free domain. It is closed there, and the Morse hypothesis makes each of its points isolated, so it is finite. Moreover, V is harmonic off the charges, and hence Lemma 3.7 shows that every critical point has index 1 or 2.

Put $E = -\nabla V$ and choose R so large that $\overline{B}_R(0)$ contains all charges and the compact set containing all critical points in its interior. Next choose $r > 0$ so small that the closed balls $\overline{B}_r(b_i)$ are disjoint, contained in $B_R(0)$, and contain no critical points. On $\Omega = B_R(0) \setminus \bigcup_i \overline{B}_r(b_i)$ the field E has no zeros on the boundary. Additivity of the Brouwer degree therefore gives

$$\sum_{p\,:\,E(p)=0} \mathrm{ind}(E,p) = \deg\Big(\tfrac{E}{|E|}\Big| * \partial B_R(0)\Big) - \sum *i = 1^n \deg\Big(\tfrac{E}{|E|}\Big|_{\partial B_r(b_i)}\Big).$$

Here $\mathrm{ind}(E,p)$ is the local Brouwer degree, and every displayed sphere carries its standard outward orientation as the boundary of its ball. Each inner sphere has the opposite orientation when regarded as a component of $\partial\Omega$; this accounts for the minus signs.

Put $Q = \sum_i q_i > 0$. Uniformly in direction as $R \to \infty$,

$$E(x) = Q\frac{x}{|x|^3} + O(|x|^{-3}),$$

so, after increasing R if necessary, the normalized field on the outer sphere is homotopic through nonvanishing maps to $u \mapsto u$ and has degree $+1$. On the sphere $x = b_i + ru$,

$$E(b_i + ru) = q_i r^{-2} u + O(1)$$

uniformly in $u \in S^2$ as $r \downarrow 0$. After decreasing r if necessary, its normalization is likewise homotopic to $u \mapsto u$ and has standard degree $+1$. Consequently $\sum_p \mathrm{ind}(E,p) = 1 - n$. At a nondegenerate zero, the local Brouwer degree is $\operatorname{sgn}\det DE(p)$. If the Morse index of p is k, then $\operatorname{sgn}\det D^2V(p) = (-1)^k$ and therefore

$$\mathrm{ind}(E,p) = \operatorname{sgn}\det(-D^2V)(p) = -(-1)^k.$$

Thus the local index of E is $+1$ at an index-1 critical point and -1 at an index-2 critical point. Consequently $m_1 - m_2 = 1 - n$, which is (45). $\square$

For the final configuration of Theorem 3.1, the selected continuations of the three families retain their indices under the small perturbation of the strengths. By Proposition 3.3, Proposition 3.11, and the index-preserving continuation in (41) of the count (40):

$$(m_1^{\mathrm{sel}}, m_2^{\mathrm{sel}}) = \underbrace{(0,3)}\, *\mathrm{seed} + \underbrace{(9,8)}\, *\mathrm{outer} + \underbrace{(4,7)}_{\text{inner}} = (13,18).$$

The outer split $(9,8)$ is (30) minus the four inner points, which consist of the origin (index 1) and the three points of radius $\rho_{p,-}$ (index 2): $10 - 1 = 9$ and $11 - 3 = 8$. Thus

$$m_2^{\mathrm{sel}} - m_1^{\mathrm{sel}} = 18 - 13 = 5 = n - 1,$$

so the selected points already account for the full index difference required by Proposition 3.16. If the complete critical set contains e_1 additional index-1 points and e_2 additional index-2 points, then the same identity gives $e_2 - e_1 = 0$. Hence $e_1 = e_2$, so any uncounted points occur in equal numbers of the two indices and the total cardinality is $31 + e_1 + e_2 = 31 + 2e_1 \geq 31$.

Remark 3.17. The scale-separation and persistence arguments, not the ledger, establish that the three selected families are disjoint. The ledger is consistent with that count but does not assert that 31 is the exact number of critical points.

4 Eight positive charges: amplification of a five-charge seed

4.1 The eight-charge theorem and its imported seed

Theorem 4.1 (Eight positive charges)**.** *There is a configuration with eight pairwise distinct sites in* $\mathbb{R}^3$ *and eight strictly positive strengths whose Coulomb potential is Morse and has a finite critical set containing at least fifty-one points. Consequently* $M_+(8) \geq 51 > 49 = (8-1)^2$.

We use the following result of Arathoon–Ball–Kvalheim [17, Theorem 1].

Theorem 4.2 (Five-charge seed)**.** *There is an* $\epsilon_0 > 0$ *such that, for every* $0 < \epsilon < \epsilon_0$, *one may place three unit charges at the vertices of an equilateral triangle of circumradius one, centered at the origin in the plane* $z = 0$, *and equal positive charges*

$$q_\epsilon = \frac{3}{4}\epsilon^3 - \frac{5}{32}\epsilon^5$$

at $\pm\epsilon e_z$. *The resulting Coulomb potential has at least twenty-four distinct nondegenerate critical points. It may have additional critical points, possibly degenerate ones.*

We use this symmetric seed before the optional generic perturbation of its strengths in [17, Theorem 1], so that its explicit local expansion is preserved. Full Morse nondegeneracy is imposed only after constructing all eight charges.

After decreasing ϵ_0, if necessary, the proof in [17, Lemmas 1–2, Remark 2, and the proof of Theorem 1] constructs, for every $0 < \epsilon < \epsilon_0$, a selected twenty-four-point family: twenty-one local points, one of which is the origin, and the continuations of the three noncentral equilibria of the triangle. We fix the twenty-three nonzero members of this family. (For the lower-bound argument alone, it would suffice to observe that at most one of any twenty-four distinct points can be the origin.) The seed expansion below also verifies directly that the origin is nondegenerate. The same two-scale mechanism as in Section 3 preserves seventeen outer local points and creates eleven innermost points. Together with twenty-three selected nonzero seed points, this gives $23 + 17 + 11 = 51$. The proof of Theorem 4.1 occupies the remaining subsections of this section. We include the scale analysis in full because its seed coefficient A depends on the previously fixed parameter ϵ; this makes the order of parameter selection and every required uniformity statement explicit.

4.2 Notation and the five-charge seed

Recall that $e_z = (0, 0, 1)$. In Cartesian coordinates (x, y, z), put $\rho^2 = x^2 + y^2$; when $\rho > 0$, let θ denote the azimuthal angle, so that $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Define

$$H_2 = \rho^2 - 2z^2, \qquad H_3 = x^3 - 3xy^2 = \rho^3\cos(3\theta),$$

and

$$H_4 = 3\rho^4 - 24\rho^2 z^2 + 8z^4.$$

These are homogeneous polynomials of degrees 2, 3, and 4, respectively, and direct differentiation gives $\Delta H_2 = \Delta H_3 = \Delta H_4 = 0$. Thus all three are harmonic. The cylindrical expression for H_3 is used only when $\rho > 0$; its Cartesian expression defines it everywhere.

The triangle vertices are

$$a_1 = (1, 0, 0), \qquad a_2 = (-1/2, \sqrt{3}/2, 0), \qquad a_3 = (-1/2, -\sqrt{3}/2, 0).$$

They satisfy $|a_j| = 1$ and $|a_j - a_k| = \sqrt{3}$ for $j \neq k$. Thus the triangle is equilateral with circumradius one (and side length $\sqrt{3}$).

Let ϵ_0 be as in Theorem 4.2, and fix $0 < \epsilon < \min\{\epsilon_0, 1\}$. Since

$$q_\epsilon = \epsilon^3 \left(\frac{3}{4} - \frac{5}{32}\epsilon^2 \right),$$

the quantity in parentheses is at least 19/32, and hence the axial strength q_ϵ is positive. Let W denote the five-charge potential

$$W(x) = \sum_{j=1}^{3} \frac{1}{|x - a_j|} + \frac{q_\epsilon}{|x - \epsilon e_z|} + \frac{q_\epsilon}{|x + \epsilon e_z|},$$

defined on $\mathbb{R}^3 \setminus \{a_1, a_2, a_3, \epsilon e_z, -\epsilon e_z\}$.

Proposition 4.3 (Expansion of the seed potential)**.** *The potential W is real analytic near the origin and has the expansion*

$$W(x) = W(0) + AH_2(x) + BH_3(x) + O_\epsilon(|x|^4), \tag{46}$$

where

$$A = \frac{5}{32}\epsilon^2 > 0, \qquad B = \frac{15}{8} > 0. \tag{47}$$

More precisely,

$$W(x) = W(0) + \frac{5}{32}\epsilon^2 H_2(x) + \frac{15}{8} H_3(x) + \left(\frac{3}{16\epsilon^2} + \frac{13}{128} \right) H_4(x) + O_\epsilon(|x|^5),$$

where $W(0) = 3 + 2q_\epsilon/\epsilon$. Define

$$\mathcal{E}_\epsilon(x) = W(x) - W(0) - AH_2(x) - BH_3(x).$$

There are constants $0 < r_\epsilon < \epsilon$ and $C_\epsilon > 0$ such that, for every multi-index α with $|\alpha| \leq 2$,

$$|D^\alpha \mathcal{E}_\epsilon(x)| \leq C_\epsilon |x|^{4-|\alpha|} \qquad (|x| < r_\epsilon).$$

Proof. Because $0 < \epsilon < 1$, the open ball $B_\epsilon(0)$ contains no charge position, so W is real analytic there. We compute its Taylor polynomial through degree four.

Write $V_\triangle(x) = \sum_{j=1}^{3} |x - a_j|^{-1}$ for the potential of the three unit charges. For any unit vector a, Taylor expansion at the origin gives

$$\frac{1}{|x - a|} = 1 + a \cdot x + \frac{1}{2}(3(a \cdot x)^2 - |x|^2)$$

$$+\frac{1}{2}(5(a\cdot x)^3-3|x|^2(a\cdot x))$$
$$+\frac{1}{8}(35(a\cdot x)^4-30|x|^2(a\cdot x)^2+3|x|^4)+O(|x|^5).$$

For the three vectors a_j, direct calculation gives

$$\sum_{j=1}^{3}a_j=0,\qquad \sum_{j=1}^{3}(a_j\cdot x)^2=\frac{3}{2}\rho^2,\qquad \sum_{j=1}^{3}(a_j\cdot x)^3=\frac{3}{4}H_3(x),$$

and

$$\sum_{j=1}^{3}(a_j\cdot x)^4=\frac{9}{8}\rho^4.$$

Substitution into the preceding Taylor formula yields

$$V_{\triangle}(x)=3+\frac{3}{4}H_2(x)+\frac{15}{8}H_3(x)+\frac{9}{64}H_4(x)+O(|x|^5).$$

For $|x|\le \epsilon/2$, the generating function for the Legendre polynomials gives the absolutely and uniformly convergent series

$$\frac{q_\epsilon}{|x-\epsilon e_z|}+\frac{q_\epsilon}{|x+\epsilon e_z|}=\frac{2q_\epsilon}{\epsilon}\sum_{m=0}^{\infty}\left(\frac{|x|}{\epsilon}\right)^{2m}P_{2m}\left(\frac{z}{|x|}\right),$$

where each product $|x|^{2m}P_{2m}(z/|x|)$ is understood as its homogeneous polynomial extension at $x=0$. Using the formulas for P_2 and P_4 gives

$$\frac{q_\epsilon}{|x-\epsilon e_z|}+\frac{q_\epsilon}{|x+\epsilon e_z|}=\frac{2q_\epsilon}{\epsilon}-\frac{q_\epsilon}{\epsilon^3}H_2(x)+\frac{q_\epsilon}{4\epsilon^5}H_4(x)+O\left(\frac{q_\epsilon|x|^6}{\epsilon^7}\right).$$

In particular, the axial pair contributes no odd-degree terms. Using the definition of q_ϵ, the combined quadratic coefficient is

$$\frac{3}{4}-\frac{q_\epsilon}{\epsilon^3}=\frac{3}{4}-\left(\frac{3}{4}-\frac{5}{32}\epsilon^2\right)=\frac{5}{32}\epsilon^2,$$

whereas the cubic coefficient remains 15/8. The combined quartic coefficient is

$$\frac{9}{64}+\frac{q_\epsilon}{4\epsilon^5}=\frac{9}{64}+\frac{3}{16\epsilon^2}-\frac{5}{128}=\frac{3}{16\epsilon^2}+\frac{13}{128}.$$

This proves both displayed expansions and the values in equation (47).

It remains to prove the derivative estimate. By the two expansions just obtained, the Taylor series of $\mathcal{E}_\epsilon$ begins in degree four. Hence, for $|\alpha|\le 2$, every derivative of $D^\alpha\mathcal{E}_\epsilon$ of order strictly less than $4-|\alpha|$ vanishes at the origin. Choose $0<r_\epsilon<\epsilon/2$. Taylor's theorem on $\overline{B}_{r_\epsilon}(0)$, applied to the finitely many derivatives with $|\alpha|\le 2$, gives a common constant C_ϵ for which

$$|D^\alpha\mathcal{E}_\epsilon(x)|\le C_\epsilon|x|^{4-|\alpha|}\qquad (|x|<r_\epsilon).$$

This completes the proof. $\square$

The expansion has no linear term, so $\nabla W(0) = 0$. Its quadratic part is $A(x^2 + y^2 - 2z^2)$, and therefore

$$D^2 W(0) = \mathrm{diag}(2A, 2A, -4A).$$

Since $A > 0$, the origin is a nondegenerate critical point of W of Morse index 1.

Finally, let $\mathcal{K} \subset \mathbb{R}^3$ be compact. For every multi-index α with $|\alpha| \le 2$, the chain rule gives

$$D_X^\alpha \Big[\eta^{-6} \mathcal{E}_\epsilon(\eta^2 X)\Big] = \eta^{2|\alpha|-6} (D^\alpha \mathcal{E}_\epsilon)(\eta^2 X).$$

For sufficiently small $\eta > 0$, one has $\eta^2 \mathcal{K} \subset B_{r_\epsilon}(0)$, and the derivative estimate above therefore yields, uniformly for $X \in \mathcal{K}$,

$$\Big| D_X^\alpha \Big[\eta^{-6} \mathcal{E}_\epsilon(\eta^2 X)\Big]\Big| \le C_\epsilon \eta^2 |X|^{4-|\alpha|}.$$

It follows that

$$\Big\|\eta^{-6} \mathcal{E}_\epsilon(\eta^2 \,\cdot)\Big\| * C^2(\mathcal{K}) = O * \epsilon, \mathcal{K}(\eta^2) \qquad \text{as } \eta \downarrow 0.$$

In every subsequent limit, ϵ is fixed first. Thus the constants implicit in $O_\epsilon(\cdot)$ and in the derivative estimates above need not be uniform as $\epsilon \downarrow 0$.

4.3 The eight charges

Fix the dimensionless normalization constants

$$\kappa = 10, \qquad \mu = \frac{1}{2}.$$

Let $\gamma > 0$ be a parameter, to be fixed sufficiently small in Sections 4.5–4.7, and put

$$d_\gamma := \frac{\kappa \gamma}{B} = \frac{16}{3}\gamma, \tag{48}$$

where the second equality uses $B = 15/8$. The choice of d_γ gives $B d_\gamma/\gamma = \kappa$. Indeed, since H_3 is homogeneous of degree three, the change of variables $X = d_\gamma Y$, followed by division by γd_γ^2, changes the coefficient of BH_3 to $Bd_\gamma/\gamma = \kappa$. Under the preceding rescaling $x = \eta^2 X$ and division by η^6, the eighth charge has coefficient $Q_{\gamma,\eta}/\eta^8$; the same inner rescaling changes this coefficient to $Q_{\gamma,\eta}/(\eta^8 \gamma d_\gamma^3)$. We therefore choose that charge so that the latter ratio equals μ. These are precisely the coefficients used in the innermost limit of Section 4.6.

For $\eta > 0$, add the following three charges to the five-charge configuration defining W:

1. At $\pm \eta e_z$, place equal charges of strength

$$p_{\gamma,\eta} = A\eta^3 - \gamma \eta^5. \tag{49}$$

2. At the axial location δe_z, where

$$\delta = d_\gamma \eta^2 = \frac{16}{3}\gamma\eta^2, \tag{50}$$

place one charge of strength

$$Q_{\gamma,\eta} = \mu\gamma d_\gamma^3 \eta^8 = \frac{2048}{27}\gamma^4 \eta^8. \tag{51}$$

Denote the resulting eight-charge potential by

$$V_{\gamma,\eta}(x) = W(x) + \frac{p_{\gamma,\eta}}{|x-\eta e_z|} + \frac{p_{\gamma,\eta}}{|x+\eta e_z|} + \frac{Q_{\gamma,\eta}}{|x-\delta e_z|}.$$

Here ϵ has already been fixed in Section 4.2. The order of parameter selection is important: first ϵ is fixed, then γ is fixed after all later smallness conditions on it have been imposed, and only then is η chosen. For fixed $\gamma > 0$,

$$p_{\gamma,\eta} = \eta^3(A - \gamma\eta^2) > 0 \quad \text{if} \quad \eta^2 < \frac{A}{\gamma},$$

while $Q_{\gamma,\eta} > 0$ automatically. In particular, every choice

$$0 < \eta < \min\left\{\epsilon, \sqrt{\frac{A}{\gamma}}, \frac{1}{d_\gamma}\right\}$$

satisfies

$$p_{\gamma,\eta} > 0, \qquad 0 < \delta = d_\gamma \eta^2 < \eta < \epsilon.$$

The five axial positions $-\epsilon e_z, -\eta e_z, \delta e_z, \eta e_z, \epsilon e_z$ are therefore distinct, and none is a triangle vertex. Together with the three positive unit strengths and the positive seed strength q_ϵ, this proves that the construction has eight distinct positions and eight positive strengths.

4.4 The first local limit

For an axial pair of strength p at $\pm\eta e_z$, write

$$U_{p,\eta}(x) = \frac{p}{|x-\eta e_z|} + \frac{p}{|x+\eta e_z|}.$$

Taylor expansion at the origin gives

$$U_{p,\eta}(x) = \frac{2p}{\eta} - \frac{p}{\eta^3}H_2(x) + \frac{p}{4\eta^5}H_4(x) + R_{p,\eta}(x). \tag{52}$$

There is an absolute constant $C > 0$ such that, for every multi-index α with $|\alpha| \le 2$,

$$|D^\alpha R_{p,\eta}(x)| \le C|p|\eta^{-7}|x|^{6-|\alpha|} \qquad (|x| \le \eta/2). \tag{53}$$

Indeed, after writing $x = \eta u$, this is Taylor's theorem on $\overline{B}_{1/2}(0)$ for the analytic even function $u \mapsto |u-e_z|^{-1} + |u+e_z|^{-1}$, followed by rescaling. Its terms of degrees zero, two, and four are precisely those displayed in equation (52); evenness makes the remainder begin in degree six.

Set

$$\widetilde{V}_{\gamma,\eta}(X) = \frac{V_{\gamma,\eta}(\eta^2 X) - V_{\gamma,\eta}(0)}{\eta^6}.$$

Using d_γ from equation (48), define

$$K_\gamma(X) = P_\gamma(X) + \frac{(2048/27)\gamma^4}{|X - d_\gamma e_z|}, \tag{54}$$

where

$$P_\gamma(X) = \gamma H_2(X) + BH_3(X) + \frac{A}{4}H_4(X). \tag{55}$$

Proposition 4.4 (First local limit)**.** *Fix $\gamma > 0$. For every compact set $\mathcal{K} \subset \mathbb{R}^3 \setminus \{d_\gamma e_z\}$, one has*

$$\left\| \widetilde{V}_{\gamma,\eta} - (K_\gamma - K_\gamma(0)) \right\| * C^2(\mathcal{K}) = O * \epsilon, \gamma, \mathcal{K}(\eta^2) \qquad \text{as } \eta \downarrow 0.$$

Proof. Because $\mathcal{K}$ is compact and does not contain $d_\gamma e_z$, all the functions below are defined on a neighborhood of $\mathcal{K}$ for sufficiently small $\eta > 0$. Indeed, the eighth charge remains at $d_\gamma e_z$, whereas the other scaled charge positions are $\eta^{-2} a_j$, $\pm\epsilon\eta^{-2} e_z$, and $\pm\eta^{-1} e_z$, all of which leave every compact set as $\eta \downarrow 0$. Define

$$\mathcal{E}_\eta^W(X) = \eta^{-6} \mathcal{E}_\epsilon(\eta^2 X).$$

By homogeneity and the estimate following Proposition 4.3,

$$\frac{W(\eta^2 X) - W(0)}{\eta^6} = A\eta^{-2} H_2(X) + BH_3(X) + \mathcal{E}^W * \eta(X), \qquad \|\mathcal{E}^W * \eta\| * C^2(\mathcal{K}) = O * \epsilon, \mathcal{K}(\eta^2).$$

Now take $p = p_{\gamma,\eta} = A\eta^3 - \gamma\eta^5$. Since $U_{p,\eta}(0) = 2p/\eta$, equation (52) gives

$$\frac{U_{p,\eta}(\eta^2 X) - U_{p,\eta}(0)}{\eta^6} - (A\eta^{-2} - \gamma) H_2(X) + \left(\frac{A}{4} - \frac{\gamma\eta^2}{4} \right) H_4(X) + \mathcal{E}^{\mathrm{pair}} * \eta(X),$$

where

$$\mathcal{E}^{\mathrm{pair}} * \eta(X) = \eta^{-6} R_{p_{\gamma,\eta},\eta}(\eta^2 X).$$

For sufficiently small η, one has $|\eta^2 X| \le \eta/2$ uniformly on $\mathcal{K}$. Hence equation (53) and the chain rule imply, for $|\alpha| \le 2$,

$$\left| D_X^\alpha \mathcal{E}^{\mathrm{pair}} * \eta(X) \right| \le C |p * \gamma, \eta| \eta^{-1} |X|^{6-|\alpha|}.$$

Since

$$\frac{|p_{\gamma,\eta}|}{\eta} \le A\eta^2 + \gamma\eta^4,$$

it follows that

$$\|\mathcal{E}^{\mathrm{pair}} * \eta\| * C^2(\mathcal{K}) = O_{\epsilon,\gamma,\mathcal{K}}(\eta^2).$$

The terms $A\eta^{-2} H_2$ therefore cancel exactly, and the sum of the seed and axial-pair contributions is

$$P_\gamma(X) - \frac{\gamma\eta^2}{4} H_4(X) + \mathcal{E}_\eta^W(X) + \mathcal{E}^{\mathrm{pair}} * \eta(X).$$

Its difference from $P * \gamma$ is $O_{\epsilon,\gamma,\mathcal{K}}(\eta^2)$ in $C^2(\mathcal{K})$.

Finally, equations (51) and (50) give the exact identity

$$\frac{Q_{\gamma,\eta}}{\eta^6} \left(\frac{1}{|\eta^2 X - \delta e_z|} - \frac{1}{|\delta e_z|} \right) = \frac{2048}{27} \gamma^4 \left(\frac{1}{|X - d_\gamma e_z|} - \frac{1}{d_\gamma} \right).$$

Because $P_\gamma(0) = 0$, the right-hand side is exactly the point-charge part of $K_\gamma(X) - K_\gamma(0)$. Combining the three contributions proves the stated estimate. $\square$

4.5 Critical points of the polynomial part

This section and Section 4.6 both compute Hessians in cylindrical coordinates, so we first record the two elementary facts that will be used in each.

Let f be smooth near a point with $\rho > 0$, and let (e_ρ, e_θ, e_z) be the orthonormal cylindrical frame there. The entries of the Hessian bilinear form $D^2 f$ in this frame are

$$\begin{aligned} &D^2 f(e_\rho, e_\rho) = \partial_{\rho\rho} f, \qquad D^2 f(e_\theta, e_\theta) = \rho^{-2}\partial_{\theta\theta} f + \rho^{-1}\partial_\rho f, \qquad D^2 f(e_z, e_z) = \partial_{zz} f, \\ &D^2 f(e_\rho, e_z) = \partial_{\rho z} f, \qquad D^2 f(e_\rho, e_\theta) = \rho^{-1}\partial_{\rho\theta} f - \rho^{-2}\partial_\theta f, \qquad D^2 f(e_\theta, e_z) = \rho^{-1}\partial_{\theta z} f. \end{aligned} \tag{56}$$

These identities follow from $\nabla f = \partial_\rho f\, e_\rho + \rho^{-1}\partial_\theta f\, e_\theta + \partial_z f\, e_z$ together with $\partial_\theta e_\rho = e_\theta$ and $\partial_\theta e_\theta = -e_\rho$. Since the frame is orthonormal, this matrix is the Cartesian Hessian written in a rotated basis; its eigenvalues, determinant, and signature are therefore those of $D^2 f$.

Lemma 4.5 (Angular decoupling)**.** *Let p be a critical point of f with $\rho > 0$, and suppose $\partial_{\rho\theta} f(p) = \partial_{\theta z} f(p) = 0$. Then the matrix (56) is block diagonal at p, with meridional block*

$$D^2_{\rho,z} f = \begin{pmatrix} \partial_{\rho\rho} f & \partial_{\rho z} f \\ \partial_{\rho z} f & \partial_{zz} f \end{pmatrix}$$

and a single angular eigenvalue $\rho^{-2}\partial_{\theta\theta} f$. In particular $\det D^2 f = \rho^{-2}\partial_{\theta\theta} f \cdot \det D^2_{\rho,z} f$.

Proof. At a critical point $\partial_\rho f = \partial_\theta f = 0$, so both correction terms in equation (56) vanish; the hypothesis annihilates the two remaining off-block entries. □

Every function to which Lemma 4.5 is applied below has the form $u(\rho, z) + c\rho^3 \cos(3\theta)$, with $c \neq 0$. For such a function $\partial_{\theta z} f \equiv 0$ and $\partial_{\rho\theta} f = -9c\rho^2 \sin(3\theta)$, which vanishes at every critical point with $\rho > 0$ because the equation $\partial_\theta f = -3c\rho^3 \sin(3\theta) = 0$ forces $\sin(3\theta) = 0$. The hypothesis of Lemma 4.5 is therefore automatic.

Lemma 4.6 (Saddle dichotomy)**.** *Let f be harmonic on an open subset of $\mathbb{R}^3$ and let p be a nondegenerate critical point of f. Then the Morse index of p is 1 or 2; it is 1 when $\det D^2 f(p) < 0$ and 2 when $\det D^2 f(p) > 0$.*

Proof. The Hessian is symmetric with vanishing trace and no zero eigenvalue, so its three real eigenvalues are nonzero and sum to zero. Hence at least one is positive and at least one is negative, so the number k of negative eigenvalues is 1 or 2. The determinant is the product of the eigenvalues, so its sign is $(-1)^k$. □

We also use the following quantitative stability lemma. Here and below, $|\cdot|$ denotes the Euclidean norm and $\|\cdot\|$ its induced operator norm.

Lemma 4.7 (Quantitative implicit function theorem)**.** *Let $r > 0$, let $\overline{B}(x_0, r) \subset \mathbb{R}^m$, and let G be a C^1 map from an open neighborhood of $\overline{B}(x_0, r)$ to $\mathbb{R}^m$. Let L be an invertible linear map, put $\nu := \|L^{-1}\|^{-1} > 0$, and assume*

$$\|DG(x) - L\| \leq \frac{\nu}{2} \quad \textit{for all } x \in \overline{B}(x_0, r), \qquad \textit{and} \qquad |G(x_0)| \leq \frac{\nu r}{2}.$$

Then G has exactly one zero x_* in $\overline{B}(x_0, r)$, it satisfies $|x_* - x_0| \leq 2|G(x_0)|/\nu$, and $DG(x_*)$ is invertible.

Proof. Put $\Phi(x) = x - L^{-1}G(x)$, so that $D\Phi(x) = L^{-1}(L - DG(x))$ and hence $\|D\Phi\| \le 1/2$ on the ball. Thus Φ is a 1/2-contraction there. Moreover, $|\Phi(x_0) - x_0| \le |G(x_0)|/\nu \le r/2$, so, for $x \in \overline{B}(x_0, r)$,

$$|\Phi(x) - x_0| \le |\Phi(x) - \Phi(x_0)| + |\Phi(x_0) - x_0| \le \tfrac{1}{2}|x - x_0| + \tfrac{r}{2} \le r.$$

Hence Φ maps the ball into itself, and the Banach fixed-point theorem supplies a unique fixed point x_*. Because L is invertible, the fixed points of Φ are exactly the zeros of G. The contraction estimate gives

$$|x_* - x_0| \le (1 - \tfrac{1}{2})^{-1}|\Phi(x_0) - x_0| \le \frac{2|G(x_0)|}{\nu}.$$

Finally,

$$\big\|L^{-1}(DG(x_*) - L)\big\| \le \frac{1}{2},$$

so $DG(x_*) = L\big(I + L^{-1}(DG(x_*) - L)\big)$ is invertible by the Neumann-series criterion. □

The polynomial P_γ is a linear combination of H_2, H_3, and H_4, hence harmonic, so Lemma 4.6 applies to it.

Proposition 4.8 (Critical points of P_γ)**.** *Let $\gamma > 0$ and assume*

$$\frac{A\gamma}{B^2} < \frac{3}{80}. \tag{57}$$

Then P_γ has exactly 21 critical points, all nondegenerate: ten of Morse index 1 and eleven of Morse index 2. As $\gamma \downarrow 0$, four lie in an $O(\gamma)$-neighborhood of the origin (the origin itself, of index 1, together with three points of index 2 and of norm asymptotic to $2\gamma/(3B)$); eight have norm $\Theta(\sqrt{\gamma})$; and nine converge to distinct nonzero limits. The minimum mutual distance among the eight points of norm $\Theta(\sqrt{\gamma})$ is $\Theta(\gamma)$, even though pairs lying on opposite sides of the plane $z = 0$ are separated by $\Theta(\sqrt{\gamma})$.

Proof. In cylindrical coordinates,

$$P_\gamma = \gamma(\rho^2 - 2z^2) + B\rho^3\cos(3\theta) + \frac{A}{4}(3\rho^4 - 24\rho^2 z^2 + 8z^4).$$

Its first derivatives are

$$\begin{aligned}
\partial_\rho P_\gamma &= \rho(2\gamma + 3B\rho\cos(3\theta) + 3A\rho^2 - 12Az^2),\\
\partial_\theta P_\gamma &= -3B\rho^3\sin(3\theta),\\
\partial_z P_\gamma &= 4z(-\gamma - 3A\rho^2 + 2Az^2).
\end{aligned}$$

When $\rho = 0$, the x- and y-derivatives vanish identically on the axis, and the remaining Cartesian critical equation is $4z(-\gamma + 2Az^2) = 0$, giving the origin and the two axial points listed below. Suppose henceforth that $\rho > 0$. Then $\sin(3\theta) = 0$, so put $s = \cos(3\theta) \in \{+1, -1\}$. If $z = 0$, the radial equation is $2\gamma + 3Bs\rho + 3A\rho^2 = 0$. It has positive solutions only when $s = -1$, in which case it is equation (58). If $z \ne 0$, the vertical equation gives

$$z^2 = \frac{\gamma}{2A} + \frac{3}{2}\rho^2.$$

Substitution in the radial equation gives $15A\rho^2 - 3Bs\rho + 4\gamma = 0$. This has positive solutions only when $s = +1$, and then it is equation (59). Thus the following list is complete.

1. The origin.

2. Two axial points:
$$\rho = 0, \qquad z = \pm\sqrt{\frac{\gamma}{2A}}.$$

3. Six planar points. Here $z = 0$, $\cos(3\theta) = -1$, and ρ is either positive root of
$$3A\rho^2 - 3B\rho + 2\gamma = 0. \tag{58}$$
Each radius supplies three angles.

4. Twelve off-planar points. Here $\cos(3\theta) = 1$, ρ is either positive root of
$$15A\rho^2 - 3B\rho + 4\gamma = 0, \tag{59}$$
and
$$z = \pm\sqrt{\frac{\gamma}{2A} + \frac{3}{2}\rho^2}.$$
Each radius supplies three angles and two signs of z.

The discriminants of equations (58) and (59) are, respectively,
$$\Delta_p = 9B^2 - 24A\gamma, \qquad \Delta_o = 9B^2 - 240A\gamma.$$
Condition (57) is exactly what is needed for $\Delta_o > 0$, and it also implies $\Delta_p > 0$. In each quadratic the sum and product of the roots are positive, and the positive discriminant makes the roots distinct. Thus both roots of each equation are positive and simple. Denote the planar roots by $0 < \rho_{p,-} < \rho_{p,+}$ and the off-planar roots by $0 < \rho_{o,-} < \rho_{o,+}$.

We now verify nondegeneracy. At the origin, $D^2P_\gamma(0) = \mathrm{diag}(2\gamma, 2\gamma, -4\gamma)$. At either axial point, using $z^2 = \gamma/(2A)$, the three Hessian eigenvalues are $(-4\gamma, -4\gamma, 8\gamma)$. At a point with $\rho > 0$ and $\sin(3\theta) = 0$, Lemma 4.5 applies, and the angular eigenvalue is $\rho^{-2}\partial_{\theta\theta}P_\gamma = -9Bs\rho \neq 0$. At a planar point, $s = -1$ and $z = 0$. There the mixed entry vanishes,
$$\partial_{\rho z}P_\gamma = \partial_\rho\big(4z(-\gamma - 3A\rho^2 + 2Az^2)\big) = -24A\rho z = 0,$$
so the meridional block is itself diagonal, and its two entries, after using equation (58), are the radial and vertical eigenvalues
$$3\rho(2A\rho - B) = \rho\,\frac{d}{d\rho}(3A\rho^2 - 3B\rho + 2\gamma) \qquad \text{and} \qquad -4\gamma - 12A\rho^2.$$
The second is negative, and the first is nonzero because the two roots of equation (58) are simple. At an off-planar point, the critical equations and $s = 1$ give
$$\partial_{\rho\rho}P_\gamma = 3\rho(B + 2A\rho), \qquad \partial_{\rho z}P_\gamma = -24A\rho z, \qquad \partial_{zz}P_\gamma = 16Az^2.$$
Consequently, the determinant of the meridional (ρ, z)-block is
$$48A\rho z^2(B - 10A\rho), \tag{60}$$
which is nonzero because
$$\frac{d}{d\rho}(15A\rho^2 - 3B\rho + 4\gamma) = -3(B - 10A\rho)$$

does not vanish at a simple root. The angular eigenvalue is also nonzero. Thus all 21 listed points are nondegenerate.

We next determine the Morse indices. Since P_γ is harmonic, Lemma 4.6 reduces this to the sign of $\det D^2 P_\gamma$, which for $\rho > 0$ factors as in Lemma 4.5. At the origin the determinant is $-16\gamma^3 < 0$ and at an axial point it is $128\gamma^3 > 0$, so the origin has index 1 and the two axial points index 2. At a planar point the angular eigenvalue $9B\rho$ is positive and the vertical eigenvalue is negative, so the sign of the determinant is opposite to that of the radial eigenvalue, hence opposite to that of $\frac{d}{d\rho}(3A\rho^2 - 3B\rho + 2\gamma)$. This derivative is negative at the smaller root $\rho_{p,-}$ and positive at the larger root $\rho_{p,+}$, so the three points of radius $\rho_{p,-}$ have index 2 and the three of radius $\rho_{p,+}$ have index 1. At an off-planar point the angular eigenvalue $-9B\rho$ is negative, so the sign of the determinant is opposite to that of equation (60), hence opposite to that of $B - 10A\rho$. The displayed derivative shows that $B - 10A\rho > 0$ at the smaller root $\rho_{o,-}$ of equation (59) and $B - 10A\rho < 0$ at the larger root $\rho_{o,+}$, so the six points of radius $\rho_{o,-}$ have index 1 and the six of radius $\rho_{o,+}$ have index 2. In total

$$m_1 = 1 + 3 + 6 = 10, \qquad m_2 = 2 + 3 + 6 = 11. \tag{61}$$

We turn to the asymptotics, keeping ϵ, and hence $A > 0$, fixed. The quadratic formulas for the two pairs of radii give

$$\rho_{p,\pm} = \frac{3B \pm \sqrt{9B^2 - 24A\gamma}}{6A}, \qquad \rho_{o,\pm} = \frac{3B \pm \sqrt{9B^2 - 240A\gamma}}{30A}.$$

Expanding these expressions at $\gamma = 0$ gives

$$\begin{aligned} \rho_{p,-} &= \frac{2\gamma}{3B} + O(\gamma^2), & \rho_{p,+} &= \frac{B}{A} - \frac{2\gamma}{3B} + O(\gamma^2), \\ \rho_{o,-} &= \frac{4\gamma}{3B} + O(\gamma^2), & \rho_{o,+} &= \frac{B}{5A} - \frac{4\gamma}{3B} + O(\gamma^2). \end{aligned}$$

The origin and the three points with radius $\rho_{p,-}$ therefore lie in an $O(\gamma)$-neighborhood of the origin; the three nonzero points have norm asymptotic to $2\gamma/(3B)$. The two axial points and the six off-planar points with radius $\rho_{o,-}$ have norm $\Theta(\sqrt{\gamma})$, because for the latter

$$z^2 = \frac{\gamma}{2A} + O(\gamma^2).$$

The remaining three planar points converge to the three points with

$$\rho = \frac{B}{A}, \qquad z = 0, \qquad \cos(3\theta) = -1,$$

whereas the six remaining off-planar points converge to the points with

$$\rho = \frac{B}{5A}, \qquad z = \pm\sqrt{\frac{3}{2}}\,\frac{B}{5A}, \qquad \cos(3\theta) = 1.$$

These nine limits are nonzero and pairwise distinct. This proves the asserted decomposition $4 + 8 + 9 = 21$.

It remains to record the mutual separations of the eight points of norm $\Theta(\sqrt{\gamma})$; these are used in Section 4.7, and the fact that their minimum separation is *smaller* than their common norm scale is what dictates the radius of the balls employed there. The eight points are the two axial points $(0, 0, \pm\sqrt{\gamma/(2A)})$ and the six off-planar points of radius $\rho_{o,-} = 4\gamma/(3B) + O(\gamma^2)$ and heights $z = \pm\sqrt{\gamma/(2A) + O(\gamma^2)}$. Thus, although each of the eight has norm $\Theta(\sqrt{\gamma})$, the six off-planar ones hug the axis, lying at horizontal distance only $\rho_{o,-} = \Theta(\gamma)$ from it. The vertical gap between an off-planar point and the axial point with the same sign of z is

$$\sqrt{\frac{\gamma}{2A} + \frac{3}{2}\rho_{o,-}^2} - \sqrt{\frac{\gamma}{2A}} = O(\gamma^{3/2}),$$

so their distance is $\left(\rho_{o,-}^2 + O(\gamma^3)\right)^{1/2} = \rho_{o,-}(1 + O(\gamma)) = \Theta(\gamma)$. Two off-planar points with the same sign of z are at distance $\sqrt{3}\,\rho_{o,-} = \Theta(\gamma)$, and every pair with opposite signs of z has vertical separation $2\sqrt{\gamma/(2A)} + O(\gamma^{3/2}) = \Theta(\sqrt{\gamma})$. Thus every pair is separated by at least a constant multiple of γ, while the same-sign pairs just described are only a constant multiple of γ apart. The minimum mutual distance among the eight is therefore $\Theta(\gamma)$. $\square$

4.6 Eleven points on the innermost scale

Recall d_γ from equation (48), and write

$$d = d_\gamma = \frac{16}{3}\gamma = \frac{\kappa\gamma}{B}, \qquad X = dY.$$

By construction,

$$\frac{Bd}{\gamma} = \kappa = 10, \qquad \frac{(2048/27)\gamma^4}{\gamma d^3} = \mu = \frac{1}{2}.$$

By homogeneity, division of equation (54) by γd^2 gives the exact identity

$$\frac{K_\gamma(dY)}{\gamma d^2} = H_2(Y) + 10H_3(Y) + \frac{64A\gamma}{9}H_4(Y) + \frac{1}{2|Y - e_z|}. \tag{62}$$

Here $Ad^2/(4\gamma) = 64A\gamma/9$; the choices of κ and μ make the other two displayed coefficients exactly 10 and 1/2, respectively. In particular, the limiting function is independent of A and B. The rescaled potentials on the left-hand side therefore converge in C^2 on every compact subset of $\mathbb{R}^3 \setminus \{e_z\}$, as $\gamma \downarrow 0$, to

$$F(Y) = H_2(Y) + 10H_3(Y) + \frac{1}{2|Y - e_z|}. \tag{63}$$

The function F is harmonic on $\mathbb{R}^3 \setminus \{e_z\}$, being the sum of two harmonic polynomials and a Newtonian kernel, so Lemma 4.6 applies to it.

Lemma 4.9 (Critical points of F)**.** *The function F has exactly 11 critical points, all nondegenerate: four of Morse index* 1 *and seven of Morse index* 2*.*

Proof. Write

$$\mathcal{R} = \sqrt{\rho^2 + (z-1)^2}.$$

In cylindrical coordinates the first derivatives of F are

$$\begin{aligned}
\partial_\rho F &= 2\rho + 30\rho^2\cos(3\theta) - \frac{\rho}{2\mathcal{R}^3},\\
\partial_\theta F &= -30\rho^3\sin(3\theta),\\
\partial_z F &= -4z - \frac{z-1}{2\mathcal{R}^3}.
\end{aligned}$$

First suppose $\rho = 0$. The transverse Cartesian derivatives vanish on the axis. There is no solution with $z > 1$, because both terms in $\partial_z F$ are then negative. For $z < 1$, the axial critical equation reduces to

$$z(1-z)^2 = \frac{1}{8}, \qquad 0 < z < 1.$$

Here the restriction $z > 0$ follows from the equation itself. The cubic factors as $8z(1-z)^2 - 1 = (2z-1)(4z^2-6z+1)$. Its roots are $1/2$ and $(3 \pm \sqrt{5})/4$; the root with the plus sign is larger than one. Thus the two admissible roots are

$$z = \frac{1}{2}, \qquad z = \frac{3-\sqrt{5}}{4}. \tag{64}$$

At an axial critical point the Hessian is diagonal. Indeed, in Cartesian coordinates $F = (x^2 + y^2 - 2z^2) + 10(x^3 - 3xy^2) + \frac{1}{2}\mathcal{R}^{-1}$; the Hessian of the cubic term is $60\left(\begin{smallmatrix} x & -y & 0 \\ -y & -x & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, which vanishes at $x = y = 0$, while the two remaining terms are invariant under rotation about the z-axis and hence have Hessians diagonal on that axis. The repeated transverse eigenvalue and the axial eigenvalue are, using $1/(2(1-z)^2) = 4z$ from the critical equation,

$$\lambda_\perp = 2 - \frac{1}{2(1-z)^3} = \frac{2(1-3z)}{1-z}, \qquad \lambda_z = -4 + \frac{1}{(1-z)^3} = \frac{4(3z-1)}{1-z}.$$

Neither root in equation (64) equals $1/3$, so both axial critical points are nondegenerate. Moreover $\det D^2F = \lambda_\perp^2 \lambda_z$ has the sign of $3z - 1$; since $\frac{1}{2} > \frac{1}{3} > (3-\sqrt{5})/4$, Lemma 4.6 gives index 2 at $z = 1/2$ and index 1 at $z = (3-\sqrt{5})/4$.

Now suppose $\rho > 0$. The angular equation gives $s = \cos(3\theta) \in \{+1, -1\}$. The radial equation is

$$\frac{1}{2\mathcal{R}^3} = 2 + 30s\rho,$$

so admissibility requires $2 + 30s\rho > 0$. Substituting this identity into the vertical equation and solving for z gives

$$z = \frac{2+30s\rho}{6+30s\rho} \tag{65}$$

and hence $1 - z = 4/(6+30s\rho)$. Admissibility makes the numerator positive, so $0 < z < 1$. The radial equation is therefore equivalent to

$$C_s(\rho) := (2+30s\rho)\left(\rho^2 + \frac{16}{(6+30s\rho)^2}\right)^{3/2} = \frac{1}{2}. \tag{66}$$

Conversely, every admissible positive root of equation (66), together with any of the three angles having the prescribed value of s, satisfies all three critical equations.

For $s = +1$, extend C_+ continuously to $\rho = 0$ and put $h = C_+^2$. A useful rational form is

$$h(\rho) = \frac{4(15\rho+1)^2(225\rho^4 + 90\rho^3 + 9\rho^2 + 4)^3}{729(5\rho+1)^6}.$$

Direct differentiation gives

$$h'(\rho) = \frac{8\rho(15\rho+1)(225\rho^4 + 90\rho^3 + 9\rho^2 + 4)^2\,\Xi(\rho)}{243(5\rho+1)^7}, \tag{67}$$

where

$$\Xi(\rho) = 22500\rho^4 + 14625\rho^3 + 3375\rho^2 + 315\rho - 191.$$

One has

$$\Xi'(\rho) = 90000\rho^3 + 43875\rho^2 + 6750\rho + 315 > 0 \qquad (\rho > 0),$$

while $\Xi(0) = -191$ and $\Xi(\rho) \to +\infty$. Thus the polynomial Ξ has exactly one positive root, which we call ρ_*; it is negative on $(0, \rho_*)$ and positive on (ρ_*, ∞). All other factors in equation (67) are positive for $\rho > 0$. Since $C_+ > 0$, the identity $h' = 2C_+C_+'$ shows that C_+ is strictly decreasing on $(0, \rho_*)$ and strictly increasing on (ρ_*, ∞). Exact substitution gives

$$C_+(0) > \frac{1}{2}, \qquad C_+(1/10) < \frac{1}{2}, \qquad C_+(1/4) > \frac{1}{2}. \tag{68}$$

The corresponding squared values are

$$\frac{256}{729}, \qquad \frac{4750104241}{21257640000}, \qquad \frac{1944701504497}{6347497291776}.$$

The first and third fractions are greater than $1/4$, while the middle fraction is less than $1/4$; as $C_+ > 0$, the comparisons (68) follow.

We now count the solutions of $C_+ = 1/2$ on $(0, \infty)$. Because ρ_* is the global minimizer, $C_+(\rho_*) \le C_+(1/10) < 1/2$; no case distinction according to the position of ρ_* relative to $1/10$ is needed. On $(0, \rho_*]$ the function C_+ decreases strictly from $C_+(0) = 16/27 > 1/2$ to $C_+(\rho_*) < 1/2$, so it attains the value $1/2$ exactly once there. On $[\rho_*, \infty)$ it increases strictly from $C_+(\rho_*) < 1/2$ to $+\infty$, since $C_+(\rho) \sim 30\rho^4$ as $\rho \to +\infty$; so it attains the value $1/2$ exactly once there as well. Thus equation (66) has exactly two roots for $s = +1$. They are on opposite sides of $1/10$, because $C_+(1/10) < 1/2$ while $C_+(0) > 1/2$ and $C_+(1/4) > 1/2$; hence one root lies in $(0, 1/10)$ and the other in $(1/10, 1/4)$. Both are simple, since C_+' vanishes only at ρ_*, where the value of C_+ is strictly below $1/2$.

For $s = -1$, admissibility is exactly $0 < \rho < 1/15$, and $C_-(\rho) = C_+(-\rho)$, where the defining expression for C_+ is positive and smooth on $(-1/15, 0)$. For the physical radius $0 < \rho < 1/15$,

$$\Xi(-\rho) < \frac{139}{9} - 191 < 0.$$

Indeed, the positive terms in $\Xi(-\rho)$ are bounded above by $22500/15^4 + 3375/15^2 = 139/9$. In equation (67), evaluated at $-\rho$, the factors $-\rho$ and $\Xi(-\rho)$ are both negative and every other factor is positive. Since $h = C_+^2$ and $C_+ > 0$ on this extended interval, $C_+'(-\rho) > 0$, and consequently $C_-'(\rho) = -C_+'(-\rho) < 0$. Moreover,

$$C_-(0) = \frac{16}{27} > \frac{1}{2}, \qquad \lim_{\rho \uparrow 1/15} C_-(\rho) = 0.$$

It follows that equation (66) has exactly one simple root for $s = -1$.

Each root with $s = +1$ supplies three angles, and the root with $s = -1$ supplies three more, giving nine off-axis points. Together with the two points in equation (64), this gives 11 critical points. At an off-axis critical point Lemma 4.5 applies, and the angular eigenvalue is $\rho^{-2}\partial_{\theta\theta}F = -90s\rho \neq 0$. We next relate the meridional Hessian determinant directly to the derivative of C_s. Put

$$a_s(\rho) = 2 + 30s\rho, \qquad q(\rho, z) = \frac{1}{2\mathcal{R}^3},$$

and define

$$G_1(\rho,z) = \frac{\partial_\rho F}{\rho} = a_s(\rho) - q(\rho,z), \qquad G_2(\rho,z) = \partial_z F = -4z - (z-1)q(\rho,z).$$

At a common zero of G_1 and G_2, differentiation of $\partial_\rho F = \rho G_1$ gives

$$\det D^2_{\rho,z} F = \rho \det D_{(\rho,z)}(G_1, G_2). \tag{69}$$

Now set

$$H = G_2 - (z-1)G_1 = -4z - (z-1)a_s(\rho).$$

On the admissible interval, the equation $H = 0$ is exactly equation (65); write its solution as $z = z_s(\rho)$ and put $\mathcal{R}_s = \mathcal{R}(\rho, z_s(\rho))$. Along this graph,

$$g_s(\rho) := G_1(\rho, z_s(\rho)) = \frac{C_s(\rho) - 1/2}{\mathcal{R}_s^3}.$$

At a root of $C_s = 1/2$, it follows that $g_s' = C_s'/\mathcal{R}_s^3$. Moreover, $H_z = -(a_s + 4)$, and implicit differentiation of $H(\rho, z_s(\rho)) = 0$ gives

$$g_s' = \frac{\det D_{(\rho,z)}(G_1, H)}{H_z} = \frac{\det D_{(\rho,z)}(G_1, G_2)}{H_z};$$

the second equality holds at the common zero because differentiating $H = G_2 - (z-1)G_1$ changes the second Jacobian row by a multiple of the first (the additional $-G_1\,dz$ term vanishes there). Finally, $G_1 = 0$ gives $\mathcal{R}_s^{-3} = 2a_s$. Substitution in equation (69) therefore yields the transparent identity

$$\det D^2_{\rho,z} F = -2\rho\, a_s(\rho)\big(a_s(\rho) + 4\big)C_s'(\rho). \tag{70}$$

On the admissible intervals, $\rho > 0$, $a_s > 0$, and $a_s + 4 > 0$. Since all three roots of equation (66) are simple, $C_s'(\rho) \neq 0$; hence the meridional determinant is nonzero, with sign opposite to that of $C_s'(\rho)$. All nine off-axis points are therefore nondegenerate.

Finally we read off the Morse indices from Lemma 4.6, using the factorization $\det D^2 F = \rho^{-2}\partial_{\theta\theta}F \cdot \det D^2_{\rho,z}F$ of Lemma 4.5. At the root with $s = +1$ lying in $(0, \rho_*)$ one has $C_+' < 0$, so the meridional determinant is positive while the angular eigenvalue -90ρ is negative; the full determinant is negative and the index is 1. At the root with $s = +1$ lying in (ρ_*, ∞) one has $C_+' > 0$, so both factors are negative and the index is 2. At the root with $s = -1$ one has $C_-' < 0$, so the meridional determinant is positive while the angular eigenvalue $+90\rho$ is positive; the index is 2. Each root contributes three points. Adding the two axial points, one of each index, gives

$$m_1 = 1 + 3 = 4, \qquad m_2 = 1 + 3 + 3 = 7, \tag{71}$$

which completes the proof. □

Let $Y_1, \ldots, Y_{11}$ be the critical points in Lemma 4.9, and set

$$F_\gamma(Y) := \frac{K_\gamma(dY)}{\gamma d^2}.$$

For each j, put $L_j = D^2 F(Y_j)$ and $\nu_j = \|L_j^{-1}\|^{-1} > 0$. Choose open Euclidean balls $U_j = B(Y_j, r_j)$ whose compact closures are pairwise disjoint and avoid e_z, so small that

$$\|D^2 F(Y) - L_j\| \le \frac{\nu_j}{4} \qquad (Y \in \overline{U}_j).$$

Equation (62) gives $F_\gamma \to F$ in C^2 on $\bigcup_j \overline{U}_j$. Hence, for every sufficiently small $\gamma > 0$,

$$\|D^2 F_\gamma(Y) - L_j\| \leq \frac{\nu_j}{2} \quad (Y \in \overline{U}_j), \qquad |\nabla F_\gamma(Y_j)| \leq \frac{\nu_j r_j}{2}$$

for all j. Lemma 4.7, applied to $G = \nabla F_\gamma$, therefore gives exactly one nondegenerate critical point $Y_j(\gamma)$ in each $\overline{U}_j$, and $Y_j(\gamma) \to Y_j$. The Hessian remains in the invertible $\nu_j/2$-neighborhood of L_j, so its Morse index is unchanged. The invertible change of variables $X = dY$ and multiplication by the positive factor γd^2 preserve critical points, nondegeneracy, and Morse index. Hence

$$X_j(\gamma) = dY_j(\gamma) = dY_j + o(d), \qquad j = 1, \dots, 11,$$

are 11 distinct nondegenerate critical points of K_γ.

4.7 The remaining 17 points and the total count

Write

$$S_\gamma(X) = \frac{(2048/27)\gamma^4}{|X - de_z|}, \qquad d = d_\gamma = \frac{16}{3}\gamma,$$

so that $K_\gamma = P_\gamma + S_\gamma$. For $j = 0, 1, 2$, differentiation of the Newtonian kernel gives

$$|D^j S_\gamma(X)| \leq C_j \gamma^4 |X - de_z|^{-1-j}.$$

Here and in the derivative estimates below, the norm of a derivative tensor is its operator norm; for a zeroth derivative it is the absolute value. Call the four critical points of P_γ lying in an $O(\gamma)$-neighborhood of the origin the *inner* ones, and the remaining 17 the *outer* ones; the latter comprise the eight of norm $\Theta(\sqrt{\gamma})$ and the nine with nonzero limits. This section shows that all 17 outer points persist in K_γ.

The persistence of the eight points of norm $\Theta(\sqrt{\gamma})$ is the delicate case, because their Hessians degenerate at rate γ while their minimum mutual separation is only $\Theta(\gamma)$. We use the quantitative implicit function theorem, Lemma 4.7, to keep the neighborhoods disjoint while controlling this degeneration.

Consider now the eight critical points of P_γ on the $O(\sqrt{\gamma})$ scale. At the two axial points, the eigenvalues divided by γ are $(-4, -4, 8)$. At each of the six off-planar points with radius $\rho_{o,-}$, the angular eigenvalue and the meridional Hessian, again divided by γ, have the limits

$$\gamma^{-1}\lambda_\theta \longrightarrow -12, \qquad \gamma^{-1} D^2_{\rho,z} P_\gamma \longrightarrow \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$

These limits follow from $\rho_{o,-} = 4\gamma/(3B) + O(\gamma^2)$ and $z^2 = \gamma/(2A) + O(\gamma^2)$. Since an orthonormal cylindrical frame is an orthogonal change of basis, the eight Cartesian Hessians divided by γ therefore have all eigenvalues bounded away from zero, uniformly for small γ. Thus there are $\gamma_0 > 0$ and $\nu_0 > 0$ such that $\|(\gamma^{-1} D^2 P_\gamma(X_0))^{-1}\|^{-1} \geq \nu_0$ at each of the eight points X_0 whenever $0 < \gamma < \gamma_0$.

By the last paragraph of the proof of Proposition 4.8, the minimum mutual distance among these eight points is $\Theta(\gamma)$, because the six off-planar points lie at horizontal distance only $\rho_{o,-} = \Theta(\gamma)$ from the axis. Consequently they admit mutually disjoint closed balls of radius $c\gamma$ once $c > 0$ is a fixed sufficiently small constant. On such a ball,

$$\|D^2 P_\gamma(X) - D^2 P_\gamma(X_0)\| \leq c\gamma \sup_{\overline{B}(X_0, c\gamma)} |D^3 P_\gamma| = O_\epsilon(c\gamma),$$

because $D^3P_\gamma = B\,D^3H_3 + (A/4)D^3H_4$, in which D^3H_3 is a constant tensor and D^3H_4 is linear in X. Because ϵ, and hence A, was fixed before γ, the points on these balls satisfy $|X| = O_\epsilon(\sqrt{\gamma})$; consequently $\|D^3P_\gamma\| = O_\epsilon(1)$, uniformly over the eight balls as $\gamma \downarrow 0$. Shrinking the fixed constant $c > 0$, with dependence on ϵ allowed, therefore makes $\|D^2P_\gamma(X) - D^2P_\gamma(X_0)\| \le \nu_0\gamma/4$ throughout the ball.

On these balls, $|X - de_z| \asymp \sqrt{\gamma}$, with comparison constants allowed to depend on the fixed ϵ, since $d = \Theta(\gamma)$. Consequently,

$$|\nabla S_\gamma| = O_\epsilon(\gamma^3), \qquad |D^2S_\gamma| = O_\epsilon(\gamma^{5/2}) = o(\gamma).$$

We apply Lemma 4.7 with $G = \nabla K_\gamma$, $m = 3$, $r = c\gamma$, and $L = D^2P_\gamma(X_0)$, so that $\nu \ge \nu_0\gamma$. The first hypothesis holds because

$$\|D^2K_\gamma(X) - L\| \le \|D^2P_\gamma(X) - D^2P_\gamma(X_0)\| + \|D^2S_\gamma(X)\| \le \frac{\nu_0\gamma}{4} + O_\epsilon(\gamma^{5/2}) \le \frac{\nu_0\gamma}{2} \le \frac{\nu}{2}$$

for all small γ. The second holds because $\nabla P_\gamma(X_0) = 0$ gives $|G(X_0)| = |\nabla S_\gamma(X_0)| = O_\epsilon(\gamma^3)$, which is at most $\nu_0\gamma \cdot c\gamma/2 \le \nu r/2$ for small γ. The lemma therefore yields exactly one critical point of K_γ in each ball, it is nondegenerate, and its displacement from X_0 is $O(\gamma^{-1}\gamma^3) = O(\gamma^2) = o(\gamma)$. As the balls are disjoint, the eight continuations are distinct. Moreover, throughout each ball the straight-line family from $D^2P_\gamma(X_0)$ to $D^2K_\gamma(X)$ remains invertible by the same norm estimate. Thus each continuation has the same Morse index as its corresponding critical point of P_γ.

The remaining nine outer critical points of P_γ converge to distinct nonzero limits. At a limiting planar point, the angular, radial, and vertical eigenvalues are, respectively,

$$\frac{9B^2}{A}, \qquad \frac{3B^2}{A}, \qquad -\frac{12B^2}{A},$$

and hence are nonzero. At a limiting off-planar point, the meridional determinant in equation (60) has the nonzero limit

$$-\frac{72B^4}{125A^2},$$

while the angular eigenvalue tends to $-9B^2/(5A)$. Thus all nine limiting Hessians are invertible.

Put $P_0 = BH_3 + (A/4)H_4$. Choose fixed pairwise disjoint neighborhoods of the nine limiting critical points of P_0 whose closures avoid the origin. On their union, $P_\gamma \to P_0$ and $\|S_\gamma\| * C^2 = O(\gamma^4)$, the latter because $de_z \to 0$. Consequently $K * \gamma = P_\gamma + S_\gamma \to P_0$ in C^2 there. The neighborhoods may be shrunk to Euclidean balls on which the Hessian of P_0 differs from its value at the center by less than one quarter of the least singular value at that center. The hypotheses of Lemma 4.7 then follow from this C^2-convergence. Thus there is exactly one nondegenerate critical point of K_γ in each ball for all sufficiently small γ. The same Hessian estimate shows that its Morse index agrees with that of the limiting critical point. We have proved that all 17 outer critical points of P_γ persist in K_γ, with unchanged Morse indices.

The 11 inner critical points found in Section 4.6 satisfy $|X| = O(d) = O(\gamma)$, whereas the preceding 17 points have distance $\asymp \sqrt{\gamma}$ or $\asymp 1$ from the origin. Thus the two families are disjoint for sufficiently small γ.

Thus K_γ has at least

$$17 + 11 = 28 \tag{72}$$

nondegenerate critical points for sufficiently small $\gamma > 0$. Choose γ small enough that this conclusion and all preceding smallness conditions hold, and fix it. Label the selected points $X_1, \ldots, X_{28}$, put $L_j = D^2K_\gamma(X_j)$ and $\nu_j = \|L_j^{-1}\|^{-1}$, and choose pairwise disjoint closed balls $\overline{B}(X_j, r_j)$ whose union avoids de_z and on which $\|D^2K_\gamma - L_j\| \le \nu_j/4$. The C^2-convergence in Proposition 4.4 implies, simultaneously on these finitely many balls, that for all sufficiently small $\eta > 0$,

$$\|D^2\widetilde{V}_{\gamma,\eta} - D^2K_\gamma\| \le \frac{\nu_j}{4} \quad \text{on } \overline{B}(X_j, r_j), \qquad |\nabla\widetilde{V}_{\gamma,\eta}(X_j)| \le \frac{\nu_j r_j}{2}.$$

Together with $\|D^2K_\gamma - L_j\| \le \nu_j/4$, the first estimate gives $\|D^2\widetilde{V}_{\gamma,\eta} - L_j\| \le \nu_j/2$ on the corresponding ball. Thus both hypotheses of Lemma 4.7, applied with $G = \nabla\widetilde{V}_{\gamma,\eta}$, hold. The lemma gives exactly one nondegenerate critical point of $\widetilde{V}_{\gamma,\eta}$ in each ball. Its Hessian is within $\nu_j/2$ of L_j; hence the straight segment between the two consists of invertible symmetric matrices and the Morse index is preserved. The invertible rescaling $x = \eta^2 X$, subtraction of a constant, and multiplication by η^{-6} preserve critical points, nondegeneracy, and Morse index, so these yield 28 distinct critical points of $V_{\gamma,\eta}$. Since their X-coordinates remain bounded, all 28 lie at distance $O(\eta^2)$ from the origin in the original variables.

The 23 selected nonzero critical points of W have positive distance from the origin. Around each such point Z_j, choose mutually disjoint closed balls $\overline{B}(Z_j, s_j)$ whose union avoids the five original charge positions and the origin. Write $M_j = D^2W(Z_j)$ and $\sigma_j = \|M_j^{-1}\|^{-1}$, and shrink the corresponding ball until $\|D^2W - M_j\| \le \sigma_j/4$ there. On the union of these balls, the potentials of the second axial pair and the eighth charge tend to zero in C^2, because their strengths are respectively $O(\eta^3)$ and $O(\eta^8)$, while their positions tend to the origin. Since $\nabla W(Z_j) = 0$, this convergence gives, for all sufficiently small η,

$$\|D^2V_{\gamma,\eta} - M_j\| \le \frac{\sigma_j}{2} \quad \text{on } \overline{B}(Z_j, s_j), \qquad |\nabla V_{\gamma,\eta}(Z_j)| \le \frac{\sigma_j s_j}{2}.$$

Lemma 4.7 therefore gives exactly one nondegenerate critical point of $V_{\gamma,\eta}$ in each ball for all sufficiently small η. The same matrix-neighborhood argument preserves the Morse index of each seed point. These 23 points remain a fixed positive distance from the origin and are consequently disjoint from the 28 local points.

Finally, choose $\eta > 0$ small enough that both persistence conclusions hold and the explicit positivity and distinctness conditions in Section 4.3 are satisfied. This respects the parameter order ϵ, then γ, then η, and produces an eight-charge configuration for which $V_{\gamma,\eta}$ has at least

$$23 + 28 = 51. \tag{73}$$

For comparison, if the eighth charge is omitted, all 21 critical points of P_γ persist and the same argument gives $23 + 21 = 44$ specified critical points for seven charges. This recovers the case $m = 2$ of the iterated axial-pair construction in [17, Proposition 1].

Remark 4.10 (Bookkeeping). It is suggestive to describe the eighth charge as replacing the four inner $O(\gamma)$-scale points of P_γ by 11 points, so that $44 - 4 + 11 = 51$. This is only a mnemonic. The proof instead supplies $23 + 17 + 11 = 51$ distinct nondegenerate critical points. It neither asserts nor requires the four inner points of P_γ to disappear. A continuation of one of them need not be distinct from the eleven already counted inner points. Only additional critical points distinct from all selected points would increase the lower bound.

4.8 Removing all remaining degeneracies

The 51 counted points are already nondegenerate and therefore isolated, but the preceding argument does not exclude additional degenerate critical points. We now remove that possibility while preserving the 51 points. Keep the eight positions fixed and label them $b_1, \ldots, b_8$, with b_1, b_2, b_3 the three triangle vertices. Let $q^{(0)}$ be the strength vector constructed above, and vary $q = (q_1, \dots, q_8)$ in the open positive orthant

$$\mathcal{P} = (0, \infty)^8.$$

On $M = \mathbb{R}^3 \setminus \{b_1, \dots, b_8\}$, consider the smooth family of electric fields

$$E(q, x) = \sum_{i=1}^{8} q_i \frac{x - b_i}{|x - b_i|^3} = -\nabla V_q(x).$$

For fixed $(q, x) \in \mathcal{P} \times M$, the derivatives with respect to the strengths are

$$\partial_{q_i} E(q, x) = \frac{x - b_i}{|x - b_i|^3}.$$

These vectors span $\mathbb{R}^3$. Indeed, if a nonzero vector v were orthogonal to all of them, then $v \cdot b_i = v \cdot x$ for every i, so all eight positions would lie in one affine plane. This is impossible: the three noncollinear points b_1, b_2, b_3 determine the plane $z = 0$, whereas every axial charge has nonzero z-coordinate. Thus the partial derivative $D_q E(q, x)$ is surjective at every (q, x); a fortiori the full derivative is surjective, and hence

$$E : \mathcal{P} \times M \longrightarrow \mathbb{R}^3$$

is a submersion.

Choose pairwise disjoint implicit-function-theorem uniqueness neighborhoods in M of the 51 constructed critical points. Applied at these points, the implicit-function theorem and continuity of the Hessian give an open neighborhood $\mathcal{U} \subset \mathcal{P}$ of $q^{(0)}$ such that every $q \in \mathcal{U}$ has a distinct nondegenerate critical point in each of the 51 neighborhoods, with the same Morse index as its parent. Since a submersion is transverse to $\{0\}$, the parametric transversality theorem [22, p. 68] implies that, for almost every $q \in \mathcal{P}$, the partial map $E(q, \cdot)$ is transverse to $\{0\}$. The exceptional set has Lebesgue measure zero and therefore contains no nonempty open subset of $\mathcal{P}$. The parameters for which $E(q, \cdot)$ is transverse to $\{0\}$ are therefore dense. We may consequently choose $q' \in \mathcal{U}$ with this property. At a zero of $E(q', \cdot)$,

$$D_x E(q', x) = -D^2 V_{q'}(x).$$

Consequently, transversality to $\{0\}$ says exactly that every critical point of $V_{q'}$ is nondegenerate, while the choice $q' \in \mathcal{U}$ preserves the 51 constructed points.

Every critical point lies in the convex hull of the charge positions. At a zero of $E(q', \cdot)$, one has

$$x = \frac{\sum_i (q' * i / |x - b_i|^3) b_i}{\sum_i q' i / |x - b_i|^3},$$

which is a convex combination because all $q' i > 0$. Choose pairwise disjoint closed balls $\overline{B} R_i(b_i)$, each containing no other charge, and set

$$C_i := \sup x \in \overline{B} * R_i(b_i) \left| \sum_{j \neq i} q'_j \frac{x - b_j}{|x - b_j|^3} \right| < \infty.$$

The term $q'_i (x - b_i)/|x - b_i|^3$ has norm $q'_i / |x - b_i|^2$. Positivity of q'_i permits a radius $0 < r_i < R_i$ with $q'_i / r_i^2 > C_i$, and then

$$\frac{q'_i}{|x - b_i|^2} \left| \sum_{j \neq i} q' * j \frac{x - b_j}{|x - b_j|^3} \right| \qquad (0 < |x - b_i| < r_i).$$

Thus $E(q', x) \neq 0$ in each punctured ball $B * r_i(b_i) \setminus \{b_i\}$. Shrinking the radii if necessary, assume that these balls are pairwise disjoint.

Let H be the convex hull of the charge positions and set

$$K = H \setminus \bigcup_{i=1}^{8} B_{r_i}(b_i).$$

The preceding two paragraphs show that every critical point belongs to the compact subset $K \subset M$. Because $E(q', \cdot)$ is continuous on K, its zero set there is closed and hence compact. Each zero is isolated because its derivative is invertible. An infinite subset of a compact metric space has an accumulation point, so this compact set of isolated zeros must be finite. The potential $V_{q'}$ is therefore Morse and has a finite number of critical points, at least 51. The positions remain the original eight distinct positions, and $q' \in \mathcal{P}$ makes all eight strengths strictly positive. This completes the proof of Theorem 4.1.

4.9 The Morse-index ledger

Every construction above produces critical points together with their Morse indices, and these indices satisfy a global constraint. We record it, both as an independent check on the count of Theorem 4.1 and because it localizes what any further critical points would have to look like.

Proposition 4.11 (Index identity)**.** *Let $n \geq 1$ and $V = \sum_{i=1}^{n} q_i |x - b_i|^{-1}$ with all $q_i > 0$ and all b_i distinct, and suppose V is Morse. Let m_1 and m_2 denote the numbers of critical points of Morse index 1 and 2. Then every critical point has index 1 or 2, both numbers are finite, and*

$$m_2 - m_1 = n - 1. \tag{74}$$

Proof. The compactness argument in Section 4.8 applies verbatim to any finite set of distinct positive charges: every critical point lies in the convex hull of the charge positions, and a neighborhood of each charge is free of critical points. Since V is Morse, its critical set is therefore compact and discrete, hence finite. Moreover, V is harmonic off the charges, so Lemma 4.6 shows that every critical point has index 1 or 2.

Put $E = -\nabla V$. Choose R so large that $\overline{B}_R(0)$ contains all charges and all critical points in its interior, and choose $r > 0$ so small that the closed balls $\overline{B}_r(b_i)$ are disjoint, contained in $B_R(0)$, and free of critical points. On the compact region $\Omega = \overline{B}_R(0) \setminus \bigcup_i B_r(b_i)$ the field E has no zeros on the boundary, so the additivity and boundary formula for Brouwer degree give

$$\sum_{p\,:\,E(p)=0} \operatorname{ind}(E, p) = \deg\left(\frac{E}{|E|}\Big| * \partial B_R(0)\right) - \sum * i = 1^n \deg\left(\frac{E}{|E|}\Big|_{\partial B_r(b_i)}\right),$$

the inner boundaries entering with the opposite sign. The mean-value theorem applied to $u \mapsto u|u|^{-3}$ gives

$$E(x) = Q\frac{x}{|x|^3} + O(|x|^{-3}), \qquad Q := \sum_i q_i > 0,$$

uniformly as $|x| \to \infty$. On $\partial B_R(0)$, the leading term has norm Q/R^2, whereas the error is $O(R^{-3})$. For all sufficiently large R, the straight-line homotopy from E to the outward radial field $Qx/|x|^3$ is therefore nowhere zero, and the outer degree is 1. Likewise, near b_i,

$$E(x) = q_i \frac{x - b_i}{|x - b_i|^3} + O(1).$$

On $\partial B_r(b_i)$, the first term has norm q_i/r^2; hence, for all sufficiently small r, the analogous straight-line homotopy is nowhere zero and the degree with the standard outward orientation of $\partial B_r(b_i)$ is 1. The boundary orientation inherited from Ω is the opposite orientation on every inner sphere, which explains the minus signs above. Hence the index sum equals $1 - n$.

At a nondegenerate zero p of E of Morse index k,

$$\operatorname{ind}(E, p) = \operatorname{sign}\det DE(p) = \operatorname{sign}\det\bigl(-D^2V(p)\bigr) = (-1)^3(-1)^k = (-1)^{k+1}.$$

Therefore $\sum_p (-1)^{k(p)+1} = 1 - n$, that is, $\sum_p (-1)^{k(p)} = n - 1$. Since $k \in \{1, 2\}$, the left-hand side is $m_2 - m_1$. □

For completeness, consider two positive charges separated by distance L. The convex combination formula in Section 4.8 places every critical point at $b_1 + te$, where $e = (b_2 - b_1)/L$ and $0 < t < L$. The field equation reduces to

$$\frac{q_1}{t^2} = \frac{q_2}{(L-t)^2},$$

which has the unique solution $t = L\sqrt{q_1}/(\sqrt{q_1} + \sqrt{q_2})$. At this point the Hessian is

$$D^2V = S(3ee^\mathsf{T} - I), \qquad S = \frac{q_1}{t^3} + \frac{q_2}{(L-t)^3} > 0.$$

Its eigenvalues are $2S, -S, -S$, so the unique critical point is nondegenerate of index 2, consistently with Proposition 4.11.

We also verify directly the indices of the three noncentral triangle points used below. By symmetry, it suffices to work on the negative x-axis and write the point as $(-t, 0, 0)$, with $0 < t < 1/2$. The x-component of the electric field there, regarded as a function of the ray parameter, is

$$f_\triangle(t) = -\frac{1}{(1+t)^2} + \frac{1-2t}{(t^2 - t + 1)^{3/2}}.$$

Set

$$h(t) = \log\left(\frac{(1-2t)(1+t)^2}{(t^2 - t + 1)^{3/2}}\right).$$

Then $f_\triangle(t) = (1+t)^{-2}(e^{h(t)} - 1)$, while

$$h'(t) = \frac{3(1 - 7t + 4t^2)}{2(1-2t)(1+t)(t^2 - t + 1)}.$$

The numerator of h' has exactly one zero in $(0, 1/2)$, namely $c = (7 - \sqrt{33})/8$. Thus h increases on $(0, c)$ and strictly decreases on $(c, 1/2)$. Since $h(0) = 0$, one has $h(c) > 0$; because $h(t) \to -\infty$ as $t \uparrow 1/2$, the decreasing branch has exactly one zero t_*. At this zero,

$$f'_\triangle(t_*) = (1 + t_*)^{-2} h'(t_*) < 0,$$

because the term obtained by differentiating $(1+t)^{-2}$ is multiplied by $e^{h(t_*)} - 1 = 0$. Reflection symmetry makes the Hessian of $V_\triangle$ diagonal at $(-t_*, 0, 0)$. Since $E = -\nabla V_\triangle$ and the coordinate on the ray is $x = -t$, the identity $f_\triangle(t) = E_x(-t, 0, 0)$ gives $f'_\triangle(t) = -\partial_x E_x = \partial_{xx} V_\triangle$. Hence $\partial_{xx} V_\triangle = f'_\triangle(t_*) < 0$. Also

$$\partial_{zz} V_\triangle = -\sum_{i=1}^{3} |(-t_*, 0, 0) - a_i|^{-3} < 0.$$

Harmonicity gives $\partial_{yy}V_\triangle = -(\partial_{xx}V_\triangle + \partial_{zz}V_\triangle) > 0$. Hence this point is nondegenerate of index 2, as are its two rotations. At the center, the expansion $V_\triangle = 3 + \frac{3}{4}H_2 + \cdots$ gives the Hessian $\frac{3}{4}\,\mathrm{diag}(2,2,-4)$, so the center has index 1.

We now assemble the ledger for the configuration of Theorem 4.1. For this we take the 23 seed points to be those produced by the construction of [17, Lemmas 1–2, Remark 2, and the proof of Theorem 1]: the continuations of the three noncentral triangle critical points together with the 20 nonzero members of the 21-point family on the scale ϵ^2. The C^2-convergence in Lemma 1 of that paper and nondegeneracy of the limiting points show that these continuations retain the Morse indices of the limiting polynomial. The proof of Proposition 4.8 uses only that the three coefficients are positive and satisfy the smallness condition (57), so the proposition applies verbatim to any polynomial $\gamma H_2 + BH_3 + \frac{A}{4}H_4$ with those properties. The 21-point limiting family of [17] is the critical set of $\frac{5}{32}H_2 + \frac{15}{8}H_3 + \frac{3}{16}H_4$. To apply the general polynomial calculation to this imported model, use the coefficient triple

$$\widehat{A} = \frac{3}{4}, \qquad \widehat{B} = \frac{15}{8}, \qquad \widehat{\gamma} = \frac{5}{32}, \qquad \text{for which} \qquad \frac{\widehat{A}\widehat{\gamma}}{\widehat{B}^2} = \frac{1}{30} < \frac{3}{80}.$$

The hatted constants belong only to this imported limiting model; they do not change the fixed value $A = 5\epsilon^2/32$ in our eight-charge construction. Equation (61) therefore applies to it: it has $m_1 = 10$ and $m_2 = 11$. Discarding the origin, which has index 1, and adjoining the three noncentral triangle points, which have index 2, the 23 seed points have $m_1 = 9$ and $m_2 = 14$. For the 17 outer points we discard from equation (61) the origin (index 1) and the three inner planar points of radius $\rho_{p,-}$ (index 2), leaving $m_1 = 9$ and $m_2 = 8$. The 11 innermost points have $m_1 = 4$ and $m_2 = 7$ by equation (71). Morse indices are unchanged by all the continuations used above: along each of them the Hessian varies continuously and remains invertible, so no eigenvalue can cross zero and the signature is constant. This includes the final perturbation from $q^{(0)}$ to q' in Section 4.8: the common implicit-function neighborhood may be taken to be a ball about $q^{(0)}$, and q' may be chosen inside that ball. Every rescaling used in the construction is of the form $g(X) = cf(sX) + C$ with $c, s > 0$, and hence

$$D^2 g(X) = cs^2 D^2 f(sX);$$

it also preserves the signature. Hence:

Selected family	m_1^{sel}	m_2^{sel}	total
23 seed points (Theorem 4.2)	9	14	23
17 outer points (Section 4.7)	9	8	17
11 innermost points (Lemma 4.9)	4	7	11
Selected total	22	29	51

The ledger closes: $29 - 22 = 7 = 8 - 1$, in exact agreement with equation (74) for $n = 8$. The same computation for the specified points in the seven-charge construction of Section 4.7 gives $m_1 = 9 + 10 = 19$ and $m_2 = 14 + 11 = 25$, and $25 - 19 = 6 = 7 - 1$; and for the 24 specified points of the five-charge seed it gives $m_1 = 10$, $m_2 = 14$, and $14 - 10 = 4 = 5 - 1$.

Two consequences deserve mention. First, the index computation provides a global consistency check on the assembled family counts: the independently computed Hessian signatures satisfy the topological identity exactly. Second, the 51 constructed points already contribute the full difference $m_2 - m_1 = 7$. Proposition 4.11 therefore implies that the numbers of any additional index-1 and index-2 critical points of $V_{q'}$ must be equal. In particular, if $e_1 \geq 0$ denotes the number of additional index-1 points, then

$$\#\,\mathrm{Crit}(V_{q'}) = 51 + 2e_1.$$

The proof does not show that $e_1 = 0$; it establishes the lower bound, not an exhaustive count.

5 Nine positive charges: a positive Kelvin design and a quintic limit

5.1 The nine-charge theorem and proof architecture

Theorem 5.1 (Nine positive charges)**.** *There are nine pairwise distinct sites and nine strictly positive strengths whose Coulomb potential has a finite critical set consisting entirely of nondegenerate points and containing at least sixty-eight points. Consequently* $M_+(9) \geq 68 > 64 = (9-1)^2$.

The proof combines a positive nine-atom Kelvin design, an exact harmonic quintic with fifty-two certified Morse critical points, and sixteen certified nonzero equilibria of the limiting Coulomb field. Full moment rank realizes the quintic by positive multiscale charge configurations, and scale separation makes the fifty-two local and sixteen remote points coexist. All certificates establish lower bounds only; no completeness assertion about either root list is required.

5.2 Kelvin moments

For each integer $\ell \geq 0$, let P_ℓ be the Legendre polynomial normalized by $P_\ell(1) = 1$, and put

$$Z_\ell(x,p) = |x|^\ell |p|^\ell P_\ell\left(\frac{x \cdot p}{|x||p|}\right). \tag{75}$$

The right side has a unique polynomial extension to all (x, p), and equation (75) is understood in this extended sense when $x = 0$ or $p = 0$. The extension is harmonic and homogeneous of degree ℓ in each variable separately.

For a site $a \neq 0$ and charge q, define its Kelvin node and Kelvin weight by

$$p = \frac{a}{|a|^2}, \qquad \beta = q|p| = \frac{q}{|a|}. \tag{76}$$

The Legendre generating function gives, for $|x| < |a|$,

$$\frac{q}{|x-a|} = \beta \sum_{\ell \geq 0} Z_\ell(x,p). \tag{77}$$

Indeed,

$$|x-a| = |a|\big(1 - 2x\cdot p + |x|^2|p|^2\big)^{1/2}, \qquad |p| = |a|^{-1},$$

so this is precisely the generating series for P_ℓ. Conversely,

$$a = \frac{p}{|p|^2}, \qquad q = \frac{\beta}{|p|}. \tag{78}$$

Thus positive Kelvin weights give positive physical charges.

Let $\mathcal{H}_\ell$ denote the $(2\ell+1)$-dimensional space of real homogeneous harmonic polynomials of degree ℓ. For nine weighted Kelvin nodes set

$$M_\ell(\beta,p)(x) = \sum_{i=1}^{9} \beta_i Z_\ell(x,p_i), \qquad F_5 = (M_1,\dots,M_5). \tag{79}$$

Equivalently, $M_\ell = 0$ if and only if

$$\sum_{i=1}^{9} \beta_i H(p_i) = 0 \quad \text{for every } H \in \mathcal{H}_\ell. \tag{80}$$

To justify the equivalence, equip the restrictions of $\mathcal{H}_\ell$ to S^2 with the $L^2(S^2)$ inner product. The spherical-harmonic addition formula says that $Z_\ell(\,\cdot\,,p)$ is, up to a nonzero constant depending only on ℓ, the reproducing kernel for evaluation at p (homogeneity extends the statement from $p \in S^2$ to arbitrary p). Pairing M_ℓ with an arbitrary $H \in \mathcal{H}_\ell$ therefore gives a nonzero constant times the left side of (80). Consequently the zonal and evaluation versions of the moment map differ by a fixed invertible block-diagonal linear map and have the same Jacobian rank. The source of F_5 has dimension $9 + 3 \cdot 9 = 36$, while its target has dimension

$$\sum_{\ell=1}^{5} (2\ell+1) = 35. \tag{81}$$

5.3 A three-orbit positive D_3 design

5.3.1 The two triangular phases

Let D_3 be generated by rotation through $2\pi/3$ about the z-axis and reflection in the xz-plane. For $\rho > 0$ and $z \in \mathbb{R}$, the two families of reflection-invariant nonaxial orbits of size three are

$$T_+(\rho,z) = \{(\rho,0,z), (-\rho/2, \sqrt{3}\rho/2, z), (-\rho/2, -\sqrt{3}\rho/2, z)\}, \tag{82}$$
$$T_-(\rho,z) = \{(\rho/2, \sqrt{3}\rho/2, z), (-\rho,0,z), (\rho/2, -\sqrt{3}\rho/2, z)\}. \tag{83}$$

If ϕ is the first azimuthal angle, write $\sigma = \cos(3\phi) \in \{+1,-1\}$ for the orbit phase. Give every node in orbit j the same weight $w_j > 0$.

In the primitive real solid-harmonic basis used in the symmetry reduction and symbolic audit, the only D_3-invariant components in degrees one through five are listed below. The right column is one third of the sum over a three-node orbit.

component	orbit sum divided by 3
$Y_{1,0}$	z
$Y_{2,0}$	$\rho^2 - 2z^2$
$Y_{3,0}$	$3\rho^2 z - 2z^3$
$Y_{3,3}$	$\sigma\rho^3$
$Y_{4,0}$	$3\rho^4 - 24\rho^2 z^2 + 8z^4$
$Y_{4,3}$	$\sigma\rho^3 z$
$Y_{5,0}$	$15\rho^4 z - 40\rho^2 z^3 + 8z^5$
$Y_{5,3}$	$\sigma\rho^3(\rho^2 - 8z^2)$

All other components vanish: rotation kills azimuthal orders not divisible by three, and reflection kills the sine components. The identities in this table and the harmonicity of every displayed polynomial are checked by exact symbolic arithmetic in the certificate package.

5.3.2 The exact seven equations

For three orbits define

$$
\begin{aligned}
M_{10} &= \sum_j w_j z_j,\\
M_{20} &= \sum_j w_j(\rho_j^2 - 2z_j^2),\\
M_{30} &= \sum_j w_j(3\rho_j^2 z_j - 2z_j^3),\\
M_{33} &= \sum_j w_j \sigma_j \rho_j^3,\\
M_{40} &= \sum_j w_j(3\rho_j^4 - 24\rho_j^2 z_j^2 + 8z_j^4),\\
M_{43} &= \sum_j w_j \sigma_j \rho_j^3 z_j.
\end{aligned}
\tag{84}
$$

The harmonic moments of degrees one through four vanish exactly when

$$M_{10} = M_{20} = M_{30} = M_{33} = M_{40} = M_{43} = 0. \tag{85}$$

For degree five put

$$
\begin{aligned}
M_{50} &= \sum_j w_j z_j(15\rho_j^4 - 40\rho_j^2 z_j^2 + 8z_j^4),\\
M_{53} &= \sum_j w_j \sigma_j \rho_j^3(\rho_j^2 - 8z_j^2).
\end{aligned}
\tag{86}
$$

The exact formula

$$Z_5(x,p) = \frac{63(x\cdot p)^5 - 70|x|^2|p|^2(x\cdot p)^3 + 15|x|^4|p|^4(x\cdot p)}{8}$$

gives the orbit decomposition

$$M_5 = \frac{3M_{50}}{64}Y_{5,0} + \frac{105M_{53}}{128}Y_{5,3}. \tag{87}$$

The desired leading block has coefficient ratio $(-445)/(-15) = 89/3$. When $M_{50} \neq 0$, (87) shows that this ratio is equivalent to

$$105M_{53} - 178M_{50} = 0. \tag{88}$$

We impose this polynomial relation; the certificate below proves $M_{50} < 0$, so the resulting degree-five block is nonzero and has the desired ratio. The equations are invariant under a common positive dilation of all nodes and under a common positive rescaling of all weights. We may therefore impose the gauges $\rho_1 = w_1 = 1$. Equations (85) and (88) are seven rational polynomial equations in

$$(\rho_2, \rho_3, z_1, z_2, z_3, w_2, w_3). \tag{89}$$

5.3.3 Classification of all phase patterns

The two $m = 3$ equations classify the nominal eight sign patterns completely.

Lemma 5.2 (Phase classification)**.** *Suppose three nonaxial orbits with positive weights satisfy all six equations in* (85)*. Their heights are distinct. After relabeling so that $z_1 < z_2 < z_3$, their phase pattern is either $(-,+,-)$ or $(+,-,+)$.*

Proof. Set $u_j = w_j \sigma_j \rho_j^3 \neq 0$. The two equations say

$$\sum_j u_j = 0, \qquad \sum_j u_j z_j = 0.$$

If exactly two heights coincide, subtracting their common height times the first equation from the second forces the third u_j to vanish. If all heights coincide, $M_{10} = 0$ forces their common value to be zero, after which $M_{20} = \sum_j w_j \rho_j^2 > 0$, a contradiction. Thus the heights are distinct. For $z_1 < z_2 < z_3$, the one-dimensional nullspace is

$$(u_1, u_2, u_3) = C(z_2 - z_3, z_3 - z_1, z_1 - z_2).$$

Its signs alternate, and $w_j \rho_j^3 > 0$, proving the claim. □

Thus the two constant sign patterns are impossible. In every mixed pattern, the orbit carrying the unique sign must have the middle height. Up to orbit permutation and global phase reversal, Lemma 5.2 exhausts all possibilities. It also gives a four-variable reduction. For example, if $z_2 - z_1 = a > 0$, $z_3 - z_2 = b > 0$, and the sorted pattern is $(+,-,+)$, then, up to a common positive factor,

$$w_1 = \frac{b}{\rho_1^3}, \qquad w_2 = \frac{a+b}{\rho_2^3}, \qquad w_3 = \frac{a}{\rho_3^3}. \tag{90}$$

If $\tilde{z}_j$ are any representatives of these height differences, then a common translation $z_j = \tilde{z}_j + t$ changes the dipole moment to

$$M_{10} = \sum_j w_j \tilde{z}_j + t \sum_j w_j.$$

Because $\sum_j w_j > 0$, the dipole equation determines the translation uniquely:

$$t = -\frac{\sum_j w_j \tilde{z}_j}{\sum_j w_j}.$$

After setting $\rho_1 = 1$ and fixing the common weight factor by $w_1 = 1$, the remaining variables are (ρ_2, ρ_3, a, b). The three unused lower-moment equations, $M_{20} = M_{30} = M_{40} = 0$, together with (88), become four equations in these four variables.

5.3.4 Certified existence and full rank

We use the unsorted labels $(\sigma_1, \sigma_2, \sigma_3) = (+,+,-)$ and $\rho_1 = w_1 = 1$.

Theorem 5.3 (Certified positive design)**.** *The seven equations* (85)*,* (88) *have a unique root in the rational box stored in the accompanying certificate* `d3_design_certificate.json`*. That box is the ℓ^∞-box of radius 10^{-70} about an exact terminating-decimal center stored there; the following table gives rounded display values of its coordinates.*

j	σ_j	ρ_j	z_j	w_j
1	$+$	1	0.2914113343279304 26	1
2	$+$	2.21187686189945216	-2.79564755504646638	0.101846287034465279
3	$-$	7.47284981413392395	-1.32710054409810034	0.00503730559014282138

All nine weights are positive, and all nine nodes are nonzero and pairwise distinct. The degree-five block is a positive scalar multiple of $-15Y5, 0 - 445Y * 5, 3$*. Moreover, the unrestricted moment map* F_5 *has rank* 35 *at the exact design.*

Certificate audit. The seven center coordinates and the radius are exact rational decimals. A 640-bit Arb calculation [24] evaluates the Krawczyk map over the entire box. The maximum relative image radius is less than $3.361 \cdot 10^{-60}$, and the contraction row-sum bound is less than $3.483 \cdot 10^{-66}$. Hence the Krawczyk image is strictly interior, and the associated Newton map is a contraction. This proves existence and uniqueness in the box by Lemma 5.7, applied with a singleton parameter set.

The design certificate gives $w_3 > 0.00503730559014282137$. Expanding its full parameter box to the nine Kelvin nodes, as is done independently by the remote-root verifier, gives

$$\min_i |p_i| > 1.04159520245380587,$$
$$\min_{i \neq j} |p_i - p_j| > 1.73205080756887729.$$

The design verifier further encloses

$$M_{50} \approx -89.05481603014603594, \qquad M_{53} \approx -150.96911669872375617.$$

At the exact root, (88) holds, so (87) has ratio $89/3$ and both coefficients are negative.

Finally, the verifier constructs a rational basis of every $\mathcal{H}_\ell$, $1 \leq \ell \leq 5$, checks harmonicity and linear independence exactly, and forms the full 35×36 Jacobian of the evaluation moment map. For the specified 35×35 submatrix J_*, its outward-rounded determinant enclosure implies the explicit bounds

$$3.020250614024624831 \cdot 10^{77} < \det J_* < 3.020250614024624834 \cdot 10^{77}$$

throughout the certified design box. In particular, this minor is nonzero at the exact design. A separate interval elimination completes all 35 pivots; the minimum certified lower bound for their absolute values is greater than 0.2539687552496252. The equivalence following (80) transfers this rank statement to the zonal map F_5. □

Write $z_0 = (\beta^0, p^0)$ for the exact design certified by Theorem 5.3.

For reference, sorting the table in Theorem 5.3 by height changes the signs to $(+, -, +)$, in agreement with Lemma 5.2.

5.4 A harmonic quintic with fifty-two Morse critical points

Use the following exact solid harmonics:

$$Y_{2,0} = x^2 + y^2 - 2z^2,$$
$$Y_{3,0} = 3x^2z + 3y^2z - 2z^3, \qquad Y_{3,3} = x^3 - 3xy^2,$$
$$Y_{4,0} = 3x^4 + 6x^2y^2 + 3y^4 - 24x^2z^2 - 24y^2z^2 + 8z^4, \qquad Y_{4,3} = x^3z - 3xy^2z,$$
$$Y_{5,0} = 15x^4z + 30x^2y^2z + 15y^4z - 40x^2z^3 - 40y^2z^3 + 8z^5,$$

$$Y_{5,3} = x^5 - 2x^3y^2 - 8x^3z^2 - 3xy^4 + 24xy^2z^2.$$

Define the integer-coefficient harmonic quintic

$$\begin{aligned} Q = 43Y_{2,0} + 613Y_{3,0} - 1545Y_{3,3} - 590Y_{4,0} - 940Y_{4,3} \\ - 15Y_{5,0} - 445Y_{5,3}. \end{aligned} \tag{91}$$

Every displayed $Y_{\ell,m}$ has zero Laplacian, so Q is harmonic; this identity is also checked exactly by the symbolic audit. Its leading block is $Q_5 = -15Y_{5,0} - 445Y_{5,3}$.

Theorem 5.4 (Certified local roots)**.** *The equation $\nabla Q = 0$ has at least 52 pairwise distinct nondegenerate real solutions. The Morse ledger of these selected solutions is*

$$m_1^{\mathrm{sel}} = 26, \qquad m_2^{\mathrm{sel}} = 26, \qquad m_0^{\mathrm{sel}} = m_3^{\mathrm{sel}} = 0.$$

Certificate audit. The certificate stores 52 exact rational boxes. Its standalone verifier uses exact fraction arithmetic only. It checks all $\binom{52}{2} = 1326$ pairwise separations, 51 strict Krawczyk inclusions, and one exact rational root. For every coordinate of every nonsingleton box, the maximum absolute displacement of a Krawczyk-image endpoint from the box center, divided by the corresponding coordinate half-width, is less than 0.976. Write the box as $X = c + R_r$. If Y denotes its rational preconditioner and $[J]$ its interval Hessian, then the defect interval $(I - Y[J])R_r$ is symmetric about zero. Its weighted radius is no larger than the preceding centered-image bound. Consequently

$$\sup_{x \in X} \|I - YD^2Q(x)\|_{\infty,r} < 0.976 < 1,$$

which is precisely the contraction hypothesis in Lemma 5.7. That lemma proves existence and uniqueness in each of the 51 Krawczyk boxes; the singleton is verified directly. On every box, the interval Hessian determinant excludes zero. Since Q is harmonic, its Hessian is real symmetric and trace-free. Such a nonsingular 3×3 matrix can have neither zero nor three negative eigenvalues, because either case would make its trace nonzero. Its determinant therefore is negative exactly at index one and positive exactly at index two. The verified ledger is $(26, 26)$. $\square$

Let $c_5 = M_5(z_0)$ be the exact degree-five block of the design. By Theorem 5.3, there is a positive scalar s with

$$c_5 = sQ_5, \qquad s = -\frac{M_{50}}{320} > 0. \tag{92}$$

Put $P = sQ$ and write its homogeneous harmonic decomposition as $P = \sum_{\ell=1}^{5} c_\ell$, where $c_1 = 0$. Scaling by $s > 0$ does not change its critical points or their Morse indices.

5.5 Sixteen remote equilibria

For Kelvin parameters $z = (\beta, p)$ define

$$W_z(x) = \sum_{i=1}^{9} \beta_i D_i(x)^{-1/2}, \qquad D_i(x) = 1 - 2x \cdot p_i + |p_i|^2|x|^2. \tag{93}$$

For the physical data in (78), one has

$$D_i(x) = |p_i|^2|x - a_i|^2, \qquad \beta_i D_i(x)^{-1/2} = \frac{q_i}{|x - a_i|}.$$

Thus W_z is exactly the Coulomb potential of those charges. With $u_i(x) = |p_i|^2 x - p_i$, its field and Hessian are

$$\nabla W_z(x) = -\sum_i \beta_i u_i D_i^{-3/2}, \tag{94}$$

$$D^2 W_z(x) = \sum_i \beta_i \left(3 u_i u_i^\top D_i^{-5/2} - |p_i|^2 I D_i^{-3/2}\right). \tag{95}$$

Theorem 5.5 (Certified remote roots). *At $z = z_0$, the potential W_{z_0} has at least 16 pairwise distinct nonzero nondegenerate equilibria, disjoint from all nine sites. The Morse ledger of these selected equilibria is*

$$m_1^{\text{sel}} = 4, \qquad m_2^{\text{sel}} = 12, \qquad m_0^{\text{sel}} = m_3^{\text{sel}} = 0.$$

Certificate audit. The exact-rational verifier expands the entire seven-dimensional certified design box to all nine weights and all twenty-seven node coordinates. For each proposed root box X and every parameter vector in the full design box B, it evaluates the parametric Krawczyk enclosure

$$K - c = -Y \nabla W_B(c) + (I - Y D^2 W_B(X))[-r, r]^3$$

with rational interval arithmetic; square roots are enclosed by exact integer arithmetic. All sixteen Krawczyk images are strictly interior, the contraction bounds are below one, and the boxes are pairwise disjoint. Lemma 5.7 therefore gives one root in each box, uniformly over B. The aggregate certified bounds are

$$\begin{aligned} \text{largest contraction bound} &< 0.2350, \\ \text{smallest strict-inclusion margin} &> 7.65 \cdot 10^{-9}, \\ \text{smallest distance to the origin} &> 0.00912372468960402, \\ \text{smallest distance to a site} &> 0.120509535120092. \end{aligned}$$

Every Hessian determinant interval excludes zero. Because W_z is harmonic away from its sites, the Hessian is trace-free there; the determinant signs therefore give four roots of index one and twelve of index two, by the same three-dimensional sign argument used in Theorem 5.4. Since the proof is uniform over B, it applies in particular to z_0, the unique exact design root in the certified box. ☐

The selected roots have signed contribution $-4 + 12 = 8 = 9 - 1$, exactly the value prescribed by the global positive-charge index identity in Proposition 4.11 for a Morse nine-charge potential. This is only a compatibility check: we have not asserted that W_{z_0} is globally Morse, and no completeness conclusion is drawn.

5.6 Positive multiscale realization

Proposition 5.6 (Multiscale coexistence). *For every sufficiently small $\varepsilon > 0$, there are nine distinct sites and nine positive charges whose Coulomb potential has 52 nondegenerate critical points in an $O(\varepsilon)$ neighborhood of the origin and 16 additional nondegenerate critical points a fixed positive distance from the origin.*

Proof. By Theorem 5.3, DF_5 is onto at the exact design z_0. Since $F_5(z_0) = (0, 0, 0, 0, c_5)$, the submersion theorem supplies a neighborhood $\mathcal{N}$ of this target value and a smooth map $\Psi : \mathcal{N} \to (\mathbb{R} \times \mathbb{R}^3)^9$ such that $\Psi(F_5(z_0)) = z_0$ and $F_5 \circ \Psi = \mathrm{id}_{\mathcal{N}}$. For $\varepsilon > 0$, set

$$t_\varepsilon = (\varepsilon^4 c_1, \varepsilon^3 c_2, \varepsilon^2 c_3, \varepsilon c_4, c_5).$$

Then $t_\varepsilon \to F_5(z_0)$. Hence, for every sufficiently small positive ε, the Kelvin parameters

$$z_\varepsilon = \Psi(t_\varepsilon) = (\beta(\varepsilon), p(\varepsilon)) \longrightarrow z_0$$

satisfy

$$F_5(z_\varepsilon) = t_\varepsilon = (\varepsilon^4 c_1, \varepsilon^3 c_2, \varepsilon^2 c_3, \varepsilon c_4, c_5). \tag{96}$$

The weights remain positive, and the nodes remain nonzero and pairwise distinct, because these are open conditions. The inverse Kelvin map $p \mapsto p/|p|^2$ is an involution on $\mathbb{R}^3 \setminus \{0\}$, so the associated physical sites are also pairwise distinct.

The section Ψ is taken in the full 36-dimensional parameter space. It need not preserve D_3 symmetry, and z_ε need not belong to the seven-dimensional design box used in the remote certificate. Below we use that certificate only at z_0; persistence under unrestricted nearby Kelvin parameters follows from nondegeneracy and C^2 convergence.

Define physical sites and charges by

$$a_i(\varepsilon) = \frac{p_i(\varepsilon)}{|p_i(\varepsilon)|^2}, \qquad q_i(\varepsilon) = \varepsilon^{-5} \frac{\beta_i(\varepsilon)}{|p_i(\varepsilon)|}. \tag{97}$$

All nine charges are positive. Let V_ε be their potential. From (77) and (96), uniformly in C^2 on each fixed compact set in the y variables,

$$\begin{aligned} V_\varepsilon(\varepsilon y) - V_\varepsilon(0) &= \sum_{\ell=1}^{5} \varepsilon^{\ell-5} M_\ell(z_\varepsilon)(y) + R_\varepsilon(y) \\ &= P(y) + R_\varepsilon(y), \qquad \|R_\varepsilon\|_{C^2(K)} \longrightarrow 0. \end{aligned} \tag{98}$$

For completeness, the normalized parameters $(\beta(\varepsilon), p(\varepsilon))$ stay in a compact parameter set. More explicitly, R_ε is the sum of the terms of degrees at least six, each multiplied by $\varepsilon^{\ell-5}$. On the region where

$$\varepsilon |y| \max_i |p_i(\varepsilon)| < \tfrac{1}{2},$$

the Legendre generating function and its first two y-derivatives converge uniformly, and the tail beginning in degree six is $O(\varepsilon)$ on K.

The 52 selected critical points of P are nondegenerate by Theorem 5.4. The C^2 convergence in (98) and Lemma 4.7, applied to the gradients on 52 sufficiently small pairwise disjoint closed balls about these points, give 52 distinct nondegenerate critical points of V_ε at

$$x_j(\varepsilon) = \varepsilon(y_j + o(1)).$$

The limiting physical sites $a_i^0 = p_i^0/|p_i^0|^2$ are nonzero, and $a_i(\varepsilon) \to a_i^0$. The local roots therefore remain disjoint from all nine sites when ε is small. To compare Hessians in the two coordinate systems, observe that $\nabla_y[V_\varepsilon(\varepsilon y)] = \varepsilon \nabla_x V_\varepsilon(\varepsilon y)$ and $D_y^2[V_\varepsilon(\varepsilon y)] = \varepsilon^2 D_x^2 V_\varepsilon(\varepsilon y)$. Thus criticality, nondegeneracy, and Morse index transfer unchanged between the y and x variables, since the two Hessians differ by a positive scalar.

For fixed physical x, (97) gives the exact identity

$$\varepsilon^5 V_\varepsilon(x) = W_{z_\varepsilon}(x). \tag{99}$$

Multiplication by ε^5 does not change critical points or Morse indices. Choose sufficiently small pairwise disjoint closed balls about the sixteen roots in Theorem 5.5 that avoid both the origin and the nine limiting sites. On these neighborhoods the right side of (99) converges in C^2 to W_{z_0}, because $z_\varepsilon \to z_0$. Lemma 4.7, again applied to the gradients, therefore preserves one nondegenerate root in each ball for sufficiently small ε. These roots remain a fixed positive distance from the origin and from the moving sites, whereas the local roots are $O(\varepsilon)$; hence the two lists are disjoint. $\square$

Proposition 5.6 produces at least

$$52 + 16 = 68 \tag{100}$$

nondegenerate critical points from nine positive charges. It remains only to make the full critical set finite and Morse. A small perturbation of the sites accomplishes this without requiring any classification of additional roots.

5.7 Finiteness of the full critical set

Fix a sufficiently small ε for which the sixty-eight points in Proposition 5.6 exist. They persist throughout an open parameter neighborhood $\mathcal{U} \subset (\mathbb{R}^3)^9$ of the nine sites, with the positive charges held fixed. Shrinking $\mathcal{U}$ if necessary, the nine sites remain pairwise distinct and the sixty-eight continuations lie in pairwise disjoint neighborhoods.

On the open incidence manifold

$$\mathcal{E} = \{(a, x) : a \in \mathcal{U},\ x \neq a_i \text{ for all } i\},$$

consider

$$G(a, x) = \nabla_x V_{a,q}(x).$$

If $d_i = x - a_i$, differentiation with respect to a_i gives

$$D_{a_i} G = q_i \left(|d_i|^{-3} I - 3|d_i|^{-5} d_i d_i^{\mathsf{T}} \right). \tag{101}$$

Its two transverse eigenvalues are $q_i|d_i|^{-3}$ and its radial eigenvalue is $-2q_i|d_i|^{-3}$. It is invertible. Hence G is a submersion.

The zero set $\mathcal{Z} = G^{-1}(0)$ is therefore a smooth manifold. Apply Sard's theorem [21, Chapter 3, §1, Theorem 1.3] to the smooth projection $\pi : \mathcal{Z} \to \mathcal{U}$. Its critical values have measure zero, so its regular values are dense in the open set $\mathcal{U}$. To identify them, observe that a tangent vector $(\dot{a}, \dot{x})$ to $\mathcal{Z}$ satisfies

$$D_a G\, \dot{a} + D_x G\, \dot{x} = 0. \tag{102}$$

If $D_x G$ is surjective, then (102) can be solved for $\dot{x}$ for every prescribed $\dot{a}$, so $D\pi$ is surjective. Conversely, $D_a G$ is surjective because even one block (101) is invertible. Given $v \in \mathbb{R}^3$, choose $\dot{a}$ with $D_a G\, \dot{a} = v$. If $D\pi$ is surjective, this $\dot{a}$ lifts to a tangent vector, and then $D_x G\, \dot{x} = -v$. Hence $D\pi$ is surjective exactly when $D_x G = D^2 V$ is surjective, equivalently invertible.

Choose a regular value $a' \in \mathcal{U}$ arbitrarily close to the configuration in Proposition 5.6, and replace a by a'. The charges have not changed and remain positive; the sites remain distinct, the sixty-eight selected roots persist, and every critical point of the perturbed potential is nondegenerate.

It remains to see that there are only finitely many. At an equilibrium, put

$$\omega_i = \frac{q_i}{|x - a_i|^3} > 0.$$

The field equation gives

$$x = \frac{\sum_i \omega_i a_i}{\sum_i \omega_i}. \tag{103}$$

Thus every equilibrium lies in the compact convex hull H of the sites. For each i, choose

$$0 < \delta_i < \frac{1}{2} \min_{k \neq i} |a_i - a_k|, \qquad C_i = \sum_{k \neq i} \frac{q_k}{(|a_i - a_k| - \delta_i)^2}.$$

If $0 < |x - a_i| < \delta_i$, the magnitude of the ith field term is $q_i/|x - a_i|^2$, while the sum of the magnitudes of all other terms is at most C_i. Choose $0 < r_i < \delta_i$ with $q_i/r_i^2 > C_i$. Then the field cannot vanish in $B(a_i, r_i) \setminus \{a_i\}$. Consequently every critical point belongs to the compact set

$$K = H \setminus \bigcup_i B(a_i, r_i),$$

which avoids all singular sites; it is compact because it is closed in H. The critical set is the zero set of the continuous field on K, and is therefore closed in K and compact. Every critical point is nondegenerate and hence isolated by the implicit-function theorem. An infinite compact set has an accumulation point, contradicting isolation; the critical set is finite.

We have produced a positive nine-charge configuration admissible in the definition of $M_+(9)$ and retaining all sixty-eight selected equilibria. This proves Theorem 5.1.

5.8 Certificate methodology and reproduction

We use the following parameter-uniform form of the Krawczyk criterion; compare [23, Chapter 8, §8.2]. Vector and matrix intervals below are Cartesian products of real intervals, and all expressions involving them are evaluated by inclusion-isotonic interval arithmetic.

Lemma 5.7 (Parameter-uniform Krawczyk criterion)**.** *Let $\mathcal{B}$ be a compact parameter box, let $U \subset \mathbb{R}^m$ be open, and let $f : \mathcal{B} \times U \to \mathbb{R}^m$ be continuous, with f_b continuously differentiable in x. Let $r = (r_1, \dots, r_m) \in (0, \infty)^m$, put*

$$R_r = \prod_{i=1}^{m} [-r_i, r_i], \qquad X = c + R_r \subset U,$$

and use the weighted norm

$$\|h\| * \infty, r = \max_i \frac{|h_i|}{r_i}, \qquad \|A\| * \infty, r = \max_i \frac{1}{r_i} \sum_j |A_{ij}| r_j.$$

Suppose $[f_{\mathcal{B}}(c)]$ contains $f_b(c)$ for every $b \in \mathcal{B}$, and $[J]$ contains $D_x f_b(x)$ for every $(b, x) \in \mathcal{B} \times X$. For a fixed real matrix Y, define

$$K = c - Y[f_{\mathcal{B}}(c)] + (I - Y[J])R_r.$$

If $K \subset \operatorname{int} X$ and

$$\sup_{b \in \mathcal{B},\, x \in X} \|I - Y D_x f_b(x)\|_{\infty, r} < 1, \tag{104}$$

then, for every $b \in \mathcal{B}$, the equation $f_b(x) = 0$ has exactly one solution in X.

Proof. Fix $b \in \mathcal{B}$ and set $T_b(x) = x - Y f_b(x)$. For $x = c + h \in X$, the integral mean-value formula gives

$$f_b(c + h) = f_b(c) + A_{b,h} h, \qquad A_{b,h} = \int_0^1 D_x f_b(c + th)\, dt.$$

The segment $c + [0, 1]h$ lies in the convex box X, so every entry of $A_{b,h}$ lies in the corresponding entry of $[J]$. Hence $T_b(X) \subset K \subset \operatorname{int} X$. Applying the same integral formula between arbitrary $x, x' \in X$ and using (104) shows that T_b is a strict contraction in the weighted supremum norm. Since X is complete, the Banach fixed-point theorem gives a unique fixed point of T_b in X.

For every $x \in X$, condition (104) and the Neumann-series criterion imply that $Y D_x f_b(x)$ is invertible. In particular, Y is invertible. Thus $T_b(x) = x$ is equivalent to $f_b(x) = 0$, which proves both existence and uniqueness of the asserted zero. $\square$

The Krawczyk checks in the nonparametric design and local certificates use the singleton *parameter* case of Lemma 5.7; the singleton *spatial* root in the local certificate is checked directly. The remote certificate takes $\mathcal{B}$ to be the full seven-dimensional design box. The proof package accompanying this manuscript is the directory `maxwell9/m9_positive/`. Its primary artifacts are:

- `agents/R4-d3-design-cert/`: the reduced design certificate, the symbolic identity audit, and the rank-35 Arb verification;
- `agents/R4-d3-local-cert/`: the exact-rational certificate for the 52 local roots;
- `agents/R4-d3-remote-cert/`: the producer and exact-rational verifier for the 16 parameter-uniform remote roots; and
- `search/d3_design_analysis_seed20260824.json`: the rational candidate centers supplied to the remote producer.

The complete directory, including all certificates and verifier source code, is publicly available at https://github.com/shivakintali/Maxwell-Lower-Bounds. The exact proof-package snapshot audited here is preserved at Git commit 3d78446a398e.

From `maxwell9/m9_positive/`, using a Python 3 environment containing `python-flint` and `SymPy`, verify the supplied certificates with

```
python3 agents/R4-d3-design-cert/verify_certificate.py \
agents/R4-d3-design-cert/d3_design_certificate.json
python3 agents/R4-d3-design-cert/symbolic_audit.py
python3 agents/R4-d3-local-cert/verify_certificate.py \
agents/R4-d3-local-cert/certificate.json
python3 agents/R4-d3-remote-cert/verify_certificate.py \
agents/R4-d3-remote-cert/remote_root_certificate.json \
--expected-count 16
```

The expected outputs certify the positive design and full rank, the symbolic orbit identities, fifty-two local roots with ledger $(26, 26)$, and sixteen remote roots with ledger $(4, 12)$. The symbolic audit writes the derived file `agents/R4-d3-design-cert/symbolic_identity_certificate.json`; none of these commands replaces a supplied design or root certificate.

To regenerate the remote certificate in a separate file, optionally run

```
python3 agents/R4-d3-remote-cert/certify_remote_roots.py \
--design agents/R4-d3-design-cert/d3_design_certificate.json \
--roots search/d3_design_analysis_seed20260824.json \
--sqrt-digits 130 --radius-min-exponent 7 --radius-max-exponent 10 \
--expected-count 16 --expected-index1 4 --expected-index2 12 \
--output remote_root_certificate.reproduced.json
python3 agents/R4-d3-remote-cert/verify_certificate.py \
remote_root_certificate.reproduced.json --expected-count 16
```

The design verifier uses outward-rounded Arb arithmetic through `python-flint`; its symbolic checks use `SymPy`. The local verifier and both remote scripts otherwise use only the Python standard library. The remote producer and verifier evaluate the field, Hessian, site distances, and parametric Krawczyk map with exact rational intervals and integer square-root enclosures. The remote verifier also checks the recorded hashes of the design certificate and candidate-center file; the design verifier must be run separately to establish the existence of the exact design. Floating-point searches are used only to choose centers and rational preconditioners; no uncertified floating-point comparison is used to establish a proof claim. The reference audit used Python 3.13.9, `python-flint` 0.9.0 with FLINT 3.6.0, and `SymPy` 1.14.0. In that environment, the printed producer command reproduces the stored remote certificate byte for byte. The same byte-for-byte reproduction was also checked with Python 3.14.6.

6 Ten positive charges: a certified sextic and a degree-forced gain

6.1 The ten-charge theorem and proof architecture

Theorem 6.1 (Ten positive charges)**.** *There are ten pairwise distinct sites and ten strictly positive strengths whose Coulomb potential has a finite critical set consisting entirely of nondegenerate points and containing at least ninety-one points. Consequently*

$$M_+(10) \geq 91 > 81 = (10-1)^2.$$

A positive ten-atom Kelvin design realizes the lower Taylor blocks of an explicit harmonic sextic. A parameter-uniform exact-rational certificate proves that the sextic has eighty-three pairwise distinct Morse critical points with ledger $(41, 42)$. Persistence transfers them to a positive ten-charge potential. The global positive-charge degree identity then forces at least eight additional critical points, giving the bound ninety-one. The Kelvin normalization is restated in this section because the sextic design uses different moment constraints and a different invariant basis from the quintic construction of Section 5.

6.2 Kelvin moments and the invariant harmonic basis

For each integer $\ell \geq 0$, let P_ℓ be the Legendre polynomial normalized by $P_\ell(1) = 1$, and set

$$Z_\ell(x,p) = |x|^\ell |p|^\ell P_\ell\left(\frac{x \cdot p}{|x||p|}\right). \tag{105}$$

The right side has a unique polynomial extension to all (x, p), and (105) is understood in this extended sense when $x = 0$ or $p = 0$. The extension is harmonic and homogeneous of degree ℓ in each argument separately.

For a nonzero site a and a charge q, define

$$p = \frac{a}{|a|^2}, \qquad \beta = q|p| = \frac{q}{|a|}. \tag{106}$$

For $|x| < |a|$, the Legendre generating function gives

$$\frac{q}{|x-a|} = \beta \sum_{\ell \geq 0} Z_\ell(x,p). \tag{107}$$

Conversely,

$$a = \frac{p}{|p|^2}, \qquad q = \frac{\beta}{|p|}. \tag{108}$$

Thus positive Kelvin weights produce positive physical charges.

For weighted Kelvin nodes $z = (\beta_i, p_i)_{i=1}^{10}$, write

$$M_\ell(z)(x) = \sum_{i=1}^{10} \beta_i Z_\ell(x, p_i). \tag{109}$$

Let $\mathcal{H}_\ell$ denote the space of real homogeneous harmonic polynomials of degree ℓ. To make the coordinate conversion explicit, choose a real basis, arranged as a column $\mathbf{H}_\ell$, and put

$$G_\ell = \frac{1}{4\pi}\int_{S^2} \mathbf{H}_\ell(u)\mathbf{H}_\ell(u)^\top \, \mathrm{d}u. \tag{110}$$

The matrix G_ℓ is positive definite. The addition theorem, with the normalization $P_\ell(1) = 1$, reads

$$Z_\ell(x,p) = \frac{1}{2\ell+1}\,\mathbf{H}_\ell(x)^\top G_\ell^{-1}\mathbf{H}_\ell(p). \tag{111}$$

Thus, if $m_\ell = \sum_i \beta_i \mathbf{H}_\ell(p_i)$, then

$$M_\ell(z)(x) = \frac{1}{2\ell+1}\mathbf{H}_\ell(x)^\top G_\ell^{-1} m_\ell. \tag{112}$$

In particular, a desired coefficient vector c_ℓ in the convention $\mathbf{H}_\ell(x)^\top c_\ell$ corresponds exactly to the evaluation moment target

$$m_\ell = (2\ell+1)G_\ell c_\ell. \tag{113}$$

Consequently $M_\ell = 0$ if and only if

$$\sum_i \beta_i H(p_i) = 0 \quad \text{for every } H \in \mathcal{H}_\ell. \tag{114}$$

Equations (112)–(113) also show that Taylor-coefficient and evaluation-moment Jacobians have the same rank.

Put $\rho^2 = x^2 + y^2$. The primitive real solid harmonics used below are

$$\begin{aligned}
&Y_{10} = z, && Y_{20} = \rho^2 - 2z^2,\\
&Y_{30} = 3\rho^2 z - 2z^3, && Y_{33} = x^3 - 3xy^2, && (115)\\
&Y_{40} = 3\rho^4 - 24\rho^2 z^2 + 8z^4, && Y_{43} = (x^3 - 3xy^2)z,\\
&Y_{50} = 15\rho^4 z - 40\rho^2 z^3 + 8z^5, && Y_{53} = (x^3 - 3xy^2)(\rho^2 - 8z^2),\\
&Y_{60} = 5\rho^6 - 90\rho^4 z^2 + 120\rho^2 z^4 - 16z^6, && Y_{63} = (x^3 - 3xy^2)z(3\rho^2 - 8z^2), && (116)\\
&Y_{66} = \mathrm{Re}(x+iy)^6.
\end{aligned}$$

Direct differentiation gives $\Delta Y_{\ell m} = 0$ in every case. Let D_3 be generated by rotation through $2\pi/3$ about the z-axis and reflection in the xz-plane. In cylindrical coordinates, the rotation retains precisely the azimuthal orders divisible by three, while the reflection retains their cosine parts and removes their sine parts. It follows that the invariant subspaces in degrees one through five have the respective bases

$$\{Y_{10}\}, \quad \{Y_{20}\}, \quad \{Y_{30}, Y_{33}\}, \quad \{Y_{40}, Y_{43}\}, \quad \{Y_{50}, Y_{53}\},$$

and the invariant subspace in degree six has basis $\{Y_{60}, Y_{63}, Y_{66}\}$. For each ℓ, extend the displayed invariant basis to a basis of $\mathcal{H}_\ell$ by adjoining a basis of its $L^2(S^2)$-orthogonal complement. With this choice, G_ℓ is block diagonal; write $G_\ell^{D_3}$ for its positive-definite invariant block. For these bases, write

$$M_{\ell m}(z) = \sum_{i=1}^{10} \beta_i Y_{\ell m}(p_i) \tag{117}$$

for the corresponding evaluation-moment coordinate. This notation is distinct from the harmonic polynomial $M_\ell(z)$ in (109); the two coordinate conventions are related by (112).

6.3 A positive ten-atom Kelvin design

6.3.1 The reduced equations

For $\rho > 0$, define the two reflection-aligned triangular orbits

$$T_+(\rho, z) = \{(\rho, 0, z), (-\rho/2, \sqrt{3}\rho/2, z), (-\rho/2, -\sqrt{3}\rho/2, z)\}, \tag{118}$$

$$T_-(\rho, z) = \{(\rho/2, \sqrt{3}\rho/2, z), (-\rho, 0, z), (\rho/2, -\sqrt{3}\rho/2, z)\}. \tag{119}$$

Their order-three phases are $\sigma = +1$ and $\sigma = -1$, respectively. More precisely, on every point of a triangle of phase σ one has

$$Y_{33} = \sigma\rho^3, \qquad Y_{43} = \sigma\rho^3 z, \qquad Y_{53} = \sigma\rho^3(\rho^2 - 8z^2).$$

Thus the sum of each of these harmonics over a triangular orbit is three times the displayed value.

We use three triangular orbits with phases

$$(\sigma_1, \sigma_2, \sigma_3) = (-1, 1, -1) \tag{120}$$

and one axial node. Introduce seven reduced variables

$$(t, \rho_2, \rho_3, m, \delta, h, A)$$

on the natural parameter domain $t > 0$, $\rho_2 > 0$, $\rho_3 > 0$, and $A > 0$, and set

$$\begin{aligned} \rho_1 &= 1, & z_1 &= m + \frac{t\delta}{1+t}, & w_1 &= 1, \\ & & z_2 &= m, & w_2 &= \frac{1+t}{\rho_2^3}, \\ & & z_3 &= m - \frac{\delta}{1+t}, & w_3 &= \frac{t}{\rho_3^3}. \end{aligned} \tag{121}$$

Every node in triangle j has Kelvin weight w_j; the axial node $(0, 0, h)$ has Kelvin weight $3A$. For the reduced equations through degree five, we divide each full ten-node evaluation moment by 3. This common nonzero factor does not affect vanishing or Jacobian rank and explains the axial contributions involving A rather than $3A$. The degree-six coefficients below use the full, undivided moment.

The parametrization makes the degree-three and degree-four tesseral moments vanish identically:

$$\sum_j w_j \sigma_j \rho_j^3 = 0, \qquad \sum_j w_j \sigma_j \rho_j^3 z_j = 0.$$

Indeed, these sums reduce respectively to $-1 + (1+t) - t$ and $-z_1 + (1+t)z_2 - tz_3$. The degree-five tesseral moment is

$$\sum_j w_j \sigma_j \rho_j^3 (\rho_j^2 - 8z_j^2),$$

which, after substitution from (121), equals $1/(1+t)$ times the left side of

$$(1+t)^2\rho_2^2 - (1+t)(1+t\rho_3^2) + 8t\delta^2 = 0. \tag{122}$$

The five zonal equations are

$$0 = \sum_j w_j z_j + Ah, \tag{123}$$

$$0 = \sum_j w_j(\rho_j^2 - 2z_j^2) - 2Ah^2, \tag{124}$$

$$0 = \sum_j w_j(3\rho_j^2 z_j - 2z_j^3) - 2Ah^3, \tag{125}$$

$$0 = \sum_j w_j(3\rho_j^4 - 24\rho_j^2 z_j^2 + 8z_j^4) + 8Ah^4, \tag{126}$$

$$0 = \sum_j w_j z_j(15\rho_j^4 - 40\rho_j^2 z_j^2 + 8z_j^4) + 8Ah^5. \tag{127}$$

Thus (122)–(127) annihilate all invariant moments through degree five. To see that no component has been omitted, let μ be the resulting weighted node measure. Since μ is D_3-invariant, its moment functional takes the same value on H and on the group average of H. That average is the projection of H onto the displayed invariant subspace. Hence the remaining, noninvariant components vanish as well.

The three degree-six basis elements in (116) are mutually orthogonal on S^2, and their normalized squared norms are

$$\frac{1}{4\pi}\int_{S^2} Y_{60}^2 = \frac{256}{13}, \qquad \frac{1}{4\pi}\int_{S^2} Y_{63}^2 = \frac{128}{1365}, \qquad \frac{1}{4\pi}\int_{S^2} Y_{66}^2 = \frac{512}{3003}.$$

Consequently (111) converts their full evaluation moments to Taylor coefficients by the respective factors 1/256, 105/128, and 231/512. Summing over each three-point orbit and using $Y_{60}(0,0,h) = -16h^6$, the possibly nonzero degree-six coefficients are therefore

$$C_{60} = -\frac{3}{256}\left[\sum_j w_j(16z_j^6 - 120z_j^4\rho_j^2 + 90z_j^2\rho_j^4 - 5\rho_j^6) + 16Ah^6\right], \tag{128}$$

$$C_{63} = \frac{315}{128}\sum_j w_j\sigma_j z_j\rho_j^3(3\rho_j^2 - 8z_j^2), \tag{129}$$

$$C_{66} = \frac{693}{512}\sum_j w_j\rho_j^6. \tag{130}$$

The seventh reduced equation is

$$77C_{60} + 6C_{66} = 0. \tag{131}$$

On the natural parameter domain above, all discarded denominators are positive. After clearing them, equations (122)–(131) are seven polynomial equations in seven variables.

6.3.2 Existence and positivity certificate

Theorem 6.2 (Certified positive design)**.** *Equations (122)–(131) have a real solution, unique in the certified rational box, for which all ten Kelvin weights are positive and all ten nodes are nonzero and pairwise distinct. At this solution*

$$M_1 = \cdots = M_5 = 0, \qquad C_{66} > 0, \qquad \frac{C_{60}}{C_{66}} = -\frac{6}{77}.$$

Exact-rational certificate audit. The design verifier interprets the full decimal strings stored in the certificate as exact rational numbers and performs all polynomial, interval, and matrix operations with rational endpoints. Its rational center is displayed here only approximately:

$$\begin{aligned}(&2.8721164291186363, 0.07761962786090543, 0.02200854398407131,\ 0.10173620063066297,\\ &0.40598639329270890, -0.52631563872291775, 8.0134684420255487)\end{aligned} \tag{132}$$

in the variable order $(t, \rho_2, \rho_3, m, \delta, h, A)$. The cube used in the proof is centered at the full stored value, not at this rounded display. On the rational cube of radius $2 \cdot 10^{-70}$, an exact Krawczyk calculation for the cleared polynomial system proves strict inclusion. The maximum centered Krawczyk ratio is less than $1.759 \cdot 10^{-50}$, the interval row-sum bound for $I - R[J]$ is less than $3.102 \cdot 10^{-62} < 1$, and the maximum exact center residual is less than $8.965 \cdot 10^{-120}$. The rational preconditioner R has nonzero determinant. Lemma 6.7, with a singleton parameter box, therefore gives a unique zero of the cleared system in the cube. This zero is nonsingular; because the cleared system has rational coefficients, its coordinates are real algebraic numbers.

On the entire cube the discarded denominators are nonzero, so this zero is a solution of the original rational equations. Interval evaluation gives the strict margins

$$\min_i \beta_i \geq 1, \qquad \min_i |p_i| > 0.0222275, \qquad \min_{i \neq j} |p_i - p_j| > 0.0381199.$$

It also gives $C_{66} > 1.35600797113862 > 0$. Equation (131) proves the stated exact ratio. The moment identities derived above justify the rational equations used in the interval calculation. ☐

For orientation, the corresponding orbit data are approximately

orbit	ρ_j	z_j	weight per node
T_-	1	0.402873888900027	1
T_+	0.0776196278609054	0.101736200630663	8280.06486046668
T_-	0.0220085439840713	−0.00311250439268230	269418.929007239
axis	0	−0.526315638722918	24.0404053260766

These decimals identify the certified algebraic solution; they are not substituted into any proof equation.

6.3.3 The restricted lower-moment submersion

Allow the three radii, three heights, three per-node weights, axial height, and reduced axial weight to vary independently. This gives the eleven source coordinates

$$(\rho_1, \rho_2, \rho_3, z_1, z_2, z_3, w_1, w_2, w_3, h, A). \tag{133}$$

The reflection-symmetric D_3-invariant moments through degree five have the eight coordinates

$$(M_{10}, M_{20}, M_{30}, M_{33}, M_{40}, M_{43}, M_{50}, M_{53}). \tag{134}$$

Theorem 6.3 (Certified lower submersion)**.** *At the exact design of Theorem 6.2, the derivative of the map from the source coordinates* (133) *to the moment coordinates* (134) *has rank eight. Consequently every sufficiently small collection of D_3-invariant Taylor blocks in degrees one through five is realized by positive, nonzero, pairwise distinct Kelvin data converging to the design.*

Exact-rational certificate audit. The verifier evaluates the full 8×11 Jacobian of the reduced moments with exact rational interval arithmetic on the lifted interval enclosure of the design cube. The columns

$$\rho_1, \rho_3, z_3, z_1, \rho_2, z_2, w_3, w_1$$

form an interval-nonsingular 8×8 minor. Let $[J]$ enclose this minor on the lifted design box, and let Y be the exact inverse of the minor evaluated at the lifted rational center. The verifier proves

$$\|I - Y[J]\|_\infty < 1.383 \cdot 10^{-59} < 1.$$

The Neumann series proves that every matrix in the interval minor is nonsingular. The target has dimension eight, so the reduced Jacobian has rank exactly eight. Multiplication by the common factor 3 recovers the full evaluation moments, and (112) transfers the rank statement to Taylor-coefficient coordinates. The submersion theorem gives a local right inverse. Positivity, nonzero nodes, and pairwise distinction persist because they are open conditions and have strict margins at the design. □

Rank eight in the invariant fixed-point space is precisely the statement required below; no unrestricted moment-rank assertion is used.

6.4 The certified harmonic sextic

Define the D_3-invariant harmonic polynomial of total degree five

$$
\begin{aligned}
Q = \frac{43}{2048}Y_{20} + \frac{613}{2048}Y_{30} - \frac{1545}{2048}Y_{33} - \frac{295}{1024}Y_{40} - \frac{235}{512}Y_{43} \\
- \frac{15}{2048}Y_{50} - \frac{445}{2048}Y_{53}.
\end{aligned}
\tag{135}
$$

Write $Q = q_2 + q_3 + q_4 + q_5$ for its homogeneous decomposition.

At the source phase (120), interval evaluation gives $C_{63} < 0 < C_{66}$. Rotate all three triangular orbits through $\pi/3$. This negates the order-three coefficient and leaves the order-zero and order-six coefficients unchanged. Dividing every Kelvin weight by the positive number C_{66} preserves positivity and gives the normalized degree-six block

$$
h_{6,\alpha} = -\frac{6}{77}Y_{60} + \alpha Y_{63} + Y_{66}, \qquad \alpha = -\frac{C_{63}}{C_{66}} > 0. \tag{136}
$$

Rotation and common positive weight scaling preserve the rank-eight statement in Theorem 6.3. Rotation by $\pi/3$ normalizes D_3 and interchanges T_+ and T_-. Thus it maps the eleven-parameter family with phases $(-,+,-)$ diffeomorphically to the family with phases $(+,-,+)$; both families are D_3-invariant. Let Φ and Φ^R denote their respective reduced lower-moment maps, let S_R be this rotation of the nodes, and let L_R be its induced invertible action on the target harmonics. Equivariance gives

$$
D\Phi^R_{S_R z} \circ DS_R = L_R \circ D\Phi_z.
$$

Likewise, if S_c multiplies every weight (including A) by $c > 0$, then $\Phi(S_c z) = c\Phi(z)$ and therefore

$$
D\Phi_{S_c z} \circ DS_c = cD\Phi_z.
$$

The maps S_R, S_c, and L_R are invertible, so neither operation changes the derivative rank. The weight scaling uses the fixed positive number $c = 1/C_{66}$ evaluated at the certified design.

The algebraic number α is retained exactly. Rational interval evaluation on the certified design box gives the following enclosure. Put

$$
\begin{aligned}
A_1 &= 12903377940679509279815148000083952, \\
A_2 &= 141078016504130370285631236177079, \\
N_\alpha &= 10^{34}A_1 + A_2.
\end{aligned}
$$

Then

$$
\alpha \in I_\alpha := \left[\frac{N_\alpha}{10^{68}}, \frac{N_\alpha + 1}{10^{68}}\right]. \tag{137}
$$

The interval has width 10^{-68} and satisfies

$$
1.2903377940679509 < \alpha < 1.2903377940679510.
$$

Proposition 6.4 (Certified sextic roots)**.** *For every $\alpha \in I_\alpha$, the harmonic polynomial*

$$P_\alpha = 100000Q + h_{6,\alpha} \tag{138}$$

has at least 83 pairwise distinct nondegenerate real critical points. The 83 certified points have selected-family Morse ledger

$$(m_1, m_2) = (41, 42).$$

Exact-rational certificate audit. The file `finiteT_parametriccertificate.json` stores 83 rational Cartesian cubes, rational preconditioners, and determinant enclosures. Candidate centers were obtained from an `msolve` computation at a rational proxy for α, but they are used only as seeds. For every cube X and the entire parameter interval I_α, the verifier recomputes the three-dimensional Krawczyk enclosure and the interval Hessian using exact rational endpoints. Write R for the cube's preconditioner and $[J]$ for its interval Hessian. Each Krawczyk image lies strictly inside its cube, and every rational preconditioner is nonsingular. The largest ratio of the maximum absolute coordinate of $K - x_0$ to the cube radius is less than $3.798 \cdot 10^{-17}$. In a given row, the defect term $(I - R[J])[-r, r]^3$ is a symmetric interval of radius

$$r \sum_j \sup |(I - R[J]) * ij|.$$

Because it is a symmetric summand of the corresponding interval coordinate of $K - x_0$, this radius is at most the maximum absolute endpoint of that coordinate. Thus every interval row sum is less than $3.798 \cdot 10^{-17} < 1$. Since each actual Jacobian belongs to $[J]$, this proves the uniform contraction bound (150). Lemma 6.7 therefore gives a unique critical point in each cube for every $\alpha \in I_\alpha$.

The minimum pairwise sup-norm separation margin is greater than $5.9589 \cdot 10^{-3}$. Thus the 83 roots are pairwise distinct. Every interval Hessian determinant excludes zero. Since P_α is harmonic, its Hessian is symmetric and trace-free; the determinant signs give 41 index-one and 42 index-two points. All assertions hold uniformly over I_α, hence at the exact algebraic value determined by the design. ☐

6.5 Realization by ten positive charges

Let $\widehat{z}_0$ be the exact design obtained from Theorem 6.2 by the rotation and positive weight normalization used in (136). Then

$$M_1(\widehat{z}_0) = \cdots = M_5(\widehat{z}_0) = 0, \qquad M_6(\widehat{z}_0) = h_{6,\alpha} \tag{139}$$

for the exact algebraic α lying in (137). The lower invariant moment map has rank eight at $\widehat{z}_0$.

Write the homogeneous decomposition of the polynomial in (138) as

$$P_\alpha = H_1 + \cdots + H_6, \qquad H_1 = 0, \quad H_\ell = 100000q_\ell \ (2 \le \ell \le 5), \quad H_6 = h_{6,\alpha}.$$

Proposition 6.5 (Positive multiscale realization)**.** *For every sufficiently small $\varepsilon > 0$, there are ten pairwise distinct sites and ten positive charges whose Coulomb potential has at least 83 pairwise distinct nondegenerate critical points.*

Proof. Work in the eleven independent coordinates (133), now with the rotated phases $(+, -, +)$, and let Φ be one third of the full evaluation-moment vector in (134). Theorem 6.3 and the preceding rank-transfer argument show that Φ is a submersion at $\widehat{z}_0$. The submersion theorem therefore gives a neighborhood B of 0 in the target and a smooth local section s such that $s(0) = \widehat{z}_0$ and $\Phi \circ s$ is the identity on B.

For $1 \le \ell \le 5$, let $c_\ell(\varepsilon)$ be the coefficient vector of $\varepsilon^{6-\ell} H_\ell$ in the displayed invariant basis, and set

$$y_\ell(\varepsilon) = \frac{1}{3}(2\ell+1) G_\ell^{D_3} c_\ell(\varepsilon).$$

The factor $1/3$ accounts for the reduced-moment convention in (123)–(127). The vectors $y_\ell(\varepsilon)$ assemble to a target $y(\varepsilon) \to 0$, so for all sufficiently small $\varepsilon > 0$ we may put $z_\varepsilon = s(y(\varepsilon))$. These D_3-invariant Kelvin parameters need not satisfy the seven-variable normalizations (121); those were used only to find the initial design. They satisfy

$$z_\varepsilon = (\beta(\varepsilon), p(\varepsilon)) \longrightarrow \widehat{z}_0$$

and, by (113), satisfy exactly

$$M_\ell(z_\varepsilon) = \varepsilon^{6-\ell} H_\ell, \qquad 1 \le \ell \le 5. \tag{140}$$

Here the orthogonal extension chosen after (116) ensures that the noninvariant target coordinates are zero. The Kelvin weights remain positive, and the nodes remain nonzero and pairwise distinct, because these are open conditions at $\widehat{z}_0$.

Define physical sites and strengths by

$$a_i(\varepsilon) = \frac{p_i(\varepsilon)}{|p_i(\varepsilon)|^2}, \qquad q_i(\varepsilon) = \varepsilon^{-6} \frac{\beta_i(\varepsilon)}{|p_i(\varepsilon)|}. \tag{141}$$

Kelvin inversion is injective on $\mathbb{R}^3 \setminus \{0\}$, so the physical sites are pairwise distinct; all strengths are strictly positive. Let V_ε be their potential. By (107) and (140), for every fixed compact set $K \subset \mathbb{R}^3$,

$$\begin{aligned} V_\varepsilon(\varepsilon X) - V_\varepsilon(0) &= \sum_{\ell \ge 1} \varepsilon^{\ell-6} M_\ell(z_\varepsilon)(X) \\ &= H_1(X) + \cdots + H_5(X) + M_6(z_\varepsilon)(X) + R_\varepsilon(X), \end{aligned} \tag{142}$$

where

$$\|R_\varepsilon\| * C^2(K) \longrightarrow 0, \qquad M_6(z * \varepsilon) \longrightarrow M_6(\widehat{z}_0) = H_6. \tag{143}$$

Indeed, the parameters stay in a compact set. For fixed K and all small ε, one has

$$\varepsilon \sup_{X \in K} |X| \max_i |p_i(\varepsilon)| < \frac{1}{2}.$$

For completeness, put

$$W_z(x) = \sum_i \beta_i (1 - 2x \cdot p_i + |x|^2 |p_i|^2)^{-1/2}.$$

On a sufficiently small fixed ball about the origin, this is smooth jointly in x and in the parameters near $\widehat{z}_0$, with uniformly bounded derivatives through order seven. Its Taylor terms are $M_\ell(z)$ by (107). Taylor's theorem, also applied to its first two derivatives, gives a uniform constant C such that

$$\left| D_x^k \left(W_z - \sum_{\ell=0}^{6} M_\ell(z) \right)(x) \right| \le C|x|^{7-k}, \qquad k = 0, 1, 2.$$

Substitution of $x = \varepsilon X$ and the chain rule multiply these bounds by ε^{k-6}. Thus the normalized tail is $O(\varepsilon)$ in $C^2(K)$, as claimed.

Thus the left side of (142) converges in $C^2(K)$ to P_α. Write X_j for the 83 selected Morse critical points from Proposition 6.4, set $L_j = D^2P_\alpha(X_j)$ and $\nu_j = \|L_j^{-1}\|^{-1} > 0$, and choose pairwise disjoint closed balls $\overline{B}(X_j, r_j)$ on which $\|D^2P_\alpha - L_j\| \le \nu_j/4$. Take K to contain their union and define $f_\varepsilon(X) = V_\varepsilon(\varepsilon X) - V_\varepsilon(0)$. The C^2 convergence and the identity $\nabla P_\alpha(X_j) = 0$ imply, for all sufficiently small ε,

$$\|D^2 f_\varepsilon - L_j\| \le \nu_j/2 \quad \text{on } \overline{B}(X_j, r_j), \qquad |\nabla f_\varepsilon(X_j)| \le \nu_j r_j/2.$$

Lemma 4.7, applied to ∇f_ε, supplies one nondegenerate critical point in each ball. The Hessian remains in the invertible $\nu_j/2$-neighborhood of L_j, so its Morse index is unchanged. The resulting points correspond to critical points of V_ε in the pairwise disjoint scaled balls because $\nabla_X f_\varepsilon(X) = \varepsilon\nabla_x V_\varepsilon(\varepsilon X)$. Moreover, $D_X^2 f_\varepsilon(X) = \varepsilon^2 D_x^2 V_\varepsilon(\varepsilon X)$, so all Morse indices are retained. ☐

Notice that the degree-six block need not remain fixed along z_ε. Continuity and (139) give exactly the convergence needed in (143); no unavailable rank-eleven assertion is being used.

6.6 Finiteness of the full critical set

Fix a sufficiently small ε for which Proposition 6.5 supplies 83 selected Morse points. They persist in an open neighborhood $\mathcal{U} \subset (\mathbb{R}^3)^{10}$ of the ordered site configuration, with the positive strengths held fixed. More explicitly, intersect the 83 implicit-function neighborhoods and then shrink the intersection so that the continuation neighborhoods remain pairwise disjoint and site-free and the ten moving sites remain pairwise distinct.

On the incidence manifold

$$\mathcal{E} = \{(a, x) : a \in \mathcal{U},\ x \ne a_i \text{ for all } i\},$$

consider

$$G(a, x) = \nabla_x V_{a,q}(x).$$

If $d_i = x - a_i$, then

$$D_{a_i} G = q_i \left(|d_i|^{-3} I - 3|d_i|^{-5} d_i d_i^{\mathsf{T}} \right). \tag{144}$$

Its two transverse eigenvalues are $q_i|d_i|^{-3}$ and its radial eigenvalue is $-2q_i|d_i|^{-3}$. Thus it is invertible, and G is a submersion.

The incidence zero set $\mathcal{Z} = G^{-1}(0)$ is a smooth manifold. Let $\pi : \mathcal{Z} \to \mathcal{U}$ be projection onto the site parameter. Sard's theorem gives a dense set of regular values of π [21, Chapter 3, §1, Theorem 1.3]. At $(a, x) \in \mathcal{Z}$, a tangent vector satisfies

$$D_a G\, \dot{a} + D_x G\, \dot{x} = 0. \tag{145}$$

If $D_x G$ is surjective, this equation can be solved for $\dot{x}$ for every prescribed $\dot{a}$, so $D\pi$ is surjective. Conversely, $D_a G$ is surjective because even one block (144) is invertible. Given $v \in \mathbb{R}^3$, choose $\dot{a}$ with $D_a G\, \dot{a} = v$. If $D\pi$ is surjective, this $\dot{a}$ lifts to a tangent vector, and (145) gives $D_x G\, \dot{x} = -v$. Therefore $D\pi$ is surjective exactly when $D_x G = D^2 V$ is surjective, equivalently invertible.

Choose a regular value $a' \in \mathcal{U}$ arbitrarily close to the configuration in Proposition 6.5, and replace a by a'. The strengths have not changed and remain positive; the sites remain distinct, the 83 selected points persist, and every critical point of the perturbed potential is nondegenerate. This final perturbation is unrestricted and need not preserve D_3 symmetry. In particular, the selected Morse ledger remains $(41, 42)$.

It remains to prove finiteness. At an equilibrium put

$$\omega_i = \frac{q_i}{|x-a_i|^3} > 0.$$

The field equation is equivalent to

$$x = \frac{\sum_i \omega_i a_i}{\sum_i \omega_i}. \tag{146}$$

Hence every equilibrium lies in the compact convex hull H of the sites. For each i, choose

$$0 < \delta_i < \frac{1}{2}\min_{k\neq i}|a_i - a_k|, \qquad C_i = \sum_{k\neq i}\frac{q_k}{(|a_i-a_k|-\delta_i)^2}.$$

If $0 < |x-a_i| < \delta_i$, the magnitude of the ith field term is $q_i/|x-a_i|^2$, whereas the sum of the magnitudes of all other terms is at most C_i. Choose $0 < r_i < \delta_i$ such that $q_i/r_i^2 > C_i$. Then the field cannot vanish in $B(a_i, r_i) \setminus \{a_i\}$. Put

$$K = H \cap \bigcap_{i=1}^{10}\{x : |x-a_i| \geq r_i\}.$$

This is a compact set avoiding the singular sites, and it contains every critical point. The field is continuous on a neighborhood of K, so its zero set is closed in K and therefore compact. Every zero is nondegenerate and hence isolated by the implicit-function theorem. If the zero set were infinite, compactness would give an accumulation point in the zero set, contradicting isolation. Hence the full critical set is finite.

6.7 The global degree and the final count

The positivity of the charges now supplies the additional points needed for the stronger lower bound.

Lemma 6.6 (Positive-charge degree identity)**.** *Let $a_1, \ldots, a_n \in \mathbb{R}^3$ be pairwise distinct, and let*

$$V(x) = \sum_{i=1}^{n}\frac{q_i}{|x-a_i|}, \qquad q_i > 0,$$

have a finite nondegenerate critical set. If m_j is the number of its critical points of Morse index j, then $m_0 = m_3 = 0$ and

$$m_2 - m_1 = n - 1. \tag{147}$$

Proof. Put $F = \nabla V$. Choose R sufficiently large that the convex hull of the sites is contained in the interior of $B_R(0)$ and the leading term in the far-field expansion below dominates its remainder on $\partial B_R(0)$. The ball then contains every critical point by (146). Near a_i, the singular term $-q_i(x-a_i)/|x-a_i|^3$ dominates the bounded sum of the remaining field terms. We may therefore choose radii $r_i > 0$ such that the closed balls $\overline{B_{r_i}(a_i)}$ are pairwise disjoint and lie in $B_R(0)$, and F has no zero whenever $0 < |x-a_i| \leq r_i$. Thus all zeros of F lie in the bounded domain

$$\Omega = B_R(0) \setminus \bigcup_{i=1}^{n}\overline{B_{r_i}(a_i)}.$$

On the outer sphere $x = Ru$, uniformly for $u \in S^2$,

$$F(Ru) = -\frac{\sum_i q_i}{R^2}u + O(R^{-3}).$$

Thus $F/|F|$ on that boundary component is homotopic to the antipodal map of S^2, whose degree is -1. On the small sphere $x = a_i + r_i u$,

$$F(a_i + r_i u) = -\frac{q_i}{r_i^2} u + O(1).$$

For sufficiently small r_i, its normalized map is again homotopic to the antipodal map: the straight-line homotopy is nonvanishing by the same dominance estimate used above. It has degree -1 when the sphere has its standard orientation as the boundary of the removed ball.

As an inner boundary component of Ω, the sphere has the opposite orientation, so its contribution is $+1$. The boundary formula for Brouwer degree therefore gives

$$\deg(F, \Omega, 0) = -1 + n = n - 1. \tag{148}$$

Because every zero is nondegenerate, the left side is the sum of $\operatorname{sign}\det D^2V$ over all critical points.

Away from the sites, V is harmonic, so $\operatorname{tr} D^2V = 0$. A nonsingular real symmetric 3×3 matrix of trace zero is neither positive definite nor negative definite. Hence every critical point has index one or two. Its determinant is negative at index one and positive at index two. Equation (148) is therefore exactly (147). □

Proof of Theorem 6.1. Apply Lemma 6.6 to the configuration constructed in the preceding subsection. Its full critical set is finite and nondegenerate. The 83 selected points have ledger $(41, 42)$ and hence signed contribution $42 - 41 = 1$. If u_1, u_2 count the remaining points of indices one and two, respectively, then

$$(42 + u_2) - (41 + u_1) = 10 - 1, \qquad u_2 - u_1 = 8.$$

Consequently $u_1 + u_2 \geq 8$. The full potential has at least $83 + 8 = 91$ critical points, all produced by ten positive charges. This proves $M_+(10) \geq 91 > 81$. □

6.8 Certificate method and reproduction

We use the following parameter-uniform form of the Krawczyk criterion; compare [23, Chapter 8, §8.2]. Vector and matrix intervals are Cartesian products of real intervals, and all interval expressions are evaluated by inclusion-isotonic arithmetic.

Lemma 6.7 (Parameter-uniform Krawczyk criterion)**.** *Let $\mathcal{B}$ be a compact parameter box, let $U \subset \mathbb{R}^m$ be open, and let $F : \mathcal{B} \times U \to \mathbb{R}^m$ be continuous, with F_b continuously differentiable in x. Let $r > 0$ and $X = x_0 + [-r, r]^m \subset U$. Suppose the interval vector $[F_{\mathcal{B}}(x_0)]$ contains $F_b(x_0)$ for every $b \in \mathcal{B}$, and the interval matrix $[J]$ contains $D_x F_b(x)$ for every $(b, x) \in \mathcal{B} \times X$. For a fixed real matrix R, define*

$$K = x_0 - R[F_{\mathcal{B}}(x_0)] + (I - R[J])[-r, r]^m. \tag{149}$$

If $K \subset \operatorname{int} X$ and

$$\sup_{b \in \mathcal{B},\, x \in X} \|I - R D_x F_b(x)\|_\infty < 1, \tag{150}$$

then, for every $b \in \mathcal{B}$, the equation $F_b(x) = 0$ has exactly one solution in X.

Proof. Fix $b \in \mathcal{B}$ and define $T_b(x) = x - RF_b(x)$. If $x = x_0 + h \in X$, the integral mean-value formula gives

$$F_b(x_0 + h) = F_b(x_0) + B_{b,h} h, \qquad B_{b,h} = \int_0^1 D_x F_b(x_0 + th)\, dt.$$

Every entry of $B_{b,h}$ belongs to the corresponding entry of $[J]$, so $T_b(X) \subset K \subset \operatorname{int} X$. Applying the same integral formula between two arbitrary points of the convex cube X, condition (150) shows that T_b is a strict contraction in the supremum norm. Since X is complete, the Banach fixed-point theorem gives a unique fixed point in X. The Neumann-series criterion also shows that $RD_xF_b(x)$ is invertible for every $x \in X$; in particular, R is invertible. Hence the unique fixed point is a zero of F_b, and every zero of F_b in X is that fixed point. □

The proof package accompanying the manuscript is the directory `maxwell10/m10_positive/`. Its proof artifacts are:

- `design_fraction_certificate.json` and `d3_rank_fraction_certificate.json` in `agents/R5-pair-rigorous/`, which certify the positive design and rank-eight minor by exact rational interval arithmetic;
- `alpha_interval_certificate.json` in the same directory, which records the exact enclosure (137) obtained from the design box;
- `PT83_T100000_msolve.json` in `agents/R4-fixed-top-sextic/`, whose rational boxes are used only as candidate centers; and
- `finiteT_parametric_certificate.json` in `agents/R5-finiteT-parametric/`, which certifies the 83 roots throughout I_α.

The complete directory, including all certificates and verifier source code, is publicly available at https://github.com/shivakintali/Maxwell-Lower-Bounds. The exact proof-package snapshot audited here is preserved at Git commit 3d78446a398e. The legacy file `design_certificate_outward_original.json` in `agents/R5-pair-rigorous/` is retained only for provenance and candidate-center discovery; no proof decision depends on its multiprecision rounding.

From `maxwell10/m10_positive/`, using a Python 3 environment containing `SymPy` and `mpmath`, run the following commands. They preserve the supplied certificates: the design and rank outputs are regenerated in a temporary directory and compared byte for byte with the packaged files.

```
python3 - <<'PY'
import json, sys
from fractions import Fraction
from pathlib import Path
from tempfile import TemporaryDirectory
sys.path.insert(0, 'agents/R5-pair-rigorous')
import certify_design_fraction as design
import certify_d3_rank_fraction as rank
from certify_pair import design_alpha_interval
with TemporaryDirectory(prefix='maxwell10-replay-') as scratch:
for verifier in (design, rank):
supplied = verifier.OUTPUT
verifier.OUTPUT = Path(scratch) / supplied.name
verifier.main()
assert verifier.OUTPUT.read_bytes() == supplied.read_bytes()
alpha = design_alpha_interval()[-1]
with open('agents/R5-pair-rigorous/alpha_interval_certificate.json') as f:
endpoints = json.load(f)['positive_phase_alpha']['exact']
assert Fraction(endpoints[0]) <= alpha.lo
```

```
assert alpha.hi <= Fraction(endpoints[1])
PY
python3 agents/R5-finiteT-parametric/certify_finiteT_parametric.py \
--verify-only
```

The reported replay used Python 3.14.6, SymPy 1.14.0, and mpmath 1.3.0. The Python block regenerates and compares the deterministic exact-rational design and rank certificates. It also independently recomputes the algebraic interval (137) from the exact design cube by the exact-rational routine `design_alpha_interval` and verifies containment in the stored interval. The final command replays all 83 Krawczyk inclusions, determinant signs, pairwise separations, and recorded source hashes. Floating-point arithmetic is used only to generate candidate centers and preconditioners; it makes no proof decision. The replay reports 83 selected Morse points with ledger $(41, 42)$, and Lemma 6.6 then gives the final lower bound 91.

AI Usage: The author designed several different configurations of charges and manually prepared an initial draft with high-level proof ideas. The author then used multiple AI tools (Claude, ChatGPT and two open-weight models with open-source harnesses) and prompted these AI tools to work out the details of the equations and proofs and identify any missing gaps. The author fixed those missing gaps and iteratively verified and expanded the proofs using AI tools. The author is solely responsible for the correctness of the proofs.

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