A Quadratic Lower Bound for Maxwell's Equilibrium Problem

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Abstract

Let $M(n)$ be the supremum of the number of critical points of a Coulomb potential generated by n point charges at distinct sites in $\mathbb{R}^3$, where the supremum is restricted to potentials with finite critical set. We prove

$$M(n) \geq \frac{n^2}{4} - n + 1 \qquad (n \geq 8,\ n \equiv 0 \pmod 4),$$

and consequently $M(n) = \Omega(n^2)$.

For $n = 2N$, with N even, our construction starts from $N/2$ symmetric pairs of axial sites and one regular N-gon of sites in the equatorial plane. The charge strengths are chosen so that a rescaling of the potential converges in C^2 to an explicitly designed harmonic polynomial. This polynomial has at least $N(N-2)+1 = (N-1)^2$ nondegenerate critical points, and these points persist for the Coulomb potential. A subsequent arbitrarily small perturbation of the charge strengths makes the entire potential Morse. The charges and their total strength remain nonzero, so estimates at the sites and at infinity, together with the Morse property, make the critical set finite. The symmetric realizing vector, and every sufficiently close perturbation of it, has axial charges of both signs when $N \geq 6$.

We also prove an index formula for the critical circles of an axisymmetric harmonic polynomial of degree m. Such a polynomial has at most $\lfloor (m-1)/2 \rfloor$ critical circles, and this bound is sharp.

1 Introduction

1.1 The problem and its motivation

Let $a_1, \ldots, a_n \in \mathbb{R}^3$ be distinct sites and let $q_1, \ldots, q_n \in \mathbb{R} \setminus \{0\}$ be their charge strengths. The associated Coulomb potential is

$$V_{a,q}(x) = \sum_{i=1}^{n} \frac{q_i}{|x - a_i|}, \qquad x \in \Omega_a := \mathbb{R}^3 \setminus \{a_1, \ldots, a_n\}. \tag{1}$$

Here and below, $|\cdot|$ denotes the Euclidean norm. The electric field is $-\nabla V_{a,q}$, so its equilibria are precisely the critical points

$$\mathrm{Crit}(V_{a,q}) := \{x \in \Omega_a : \nabla V_{a,q}(x) = 0\}.$$

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Maxwell's problem asks how many such equilibria can be created by n point charges. It is a natural meeting point of classical potential theory, real algebraic geometry, and singularity theory. The potential is harmonic,

$$\Delta V_{a,q} = 0 \qquad \text{on } \Omega_a,$$

but the critical-point equations have nonpolynomial denominators, and their global complexity is not apparent from this local harmonicity.

The strong maximum principle shows that $V_{a,q}$ has no local maximum or minimum in Ω_a. In particular, every nondegenerate equilibrium is a saddle: its Hessian is nonsingular and, by harmonicity, has trace zero. Degenerate equilibria are substantially more delicate. They need not persist under perturbation, and symmetry can make the critical locus positive-dimensional. A counting problem must therefore specify both the permitted signs of the charges and the finiteness or nondegeneracy convention.

1.2 Counting conventions

A critical point x is *nondegenerate* if $\det \operatorname{Hess} V_{a,q}(x) \neq 0$; a potential all of whose critical points are nondegenerate is called *Morse*. For each positive integer n, we study

$$M(n) = \sup\bigl\{\# \operatorname{Crit}(V_{a,q}) : a_i \neq a_j \ (i \neq j),\ q_i \neq 0,\ \operatorname{Crit}(V_{a,q}) \text{ is finite}\bigr\}. \tag{2}$$

Here the supremum ranges over all choices of pairwise distinct sites and all nonzero real charge strengths. Thus $M(n)$ permits both signs and excludes every configuration with an infinite critical set—in particular, one with a critical curve—but it permits finitely many degenerate critical points. For comparison, define

$$M_+(n) = \sup\bigl\{\# \operatorname{Crit}(V_{a,q}) : a_i \neq a_j \ (i \neq j),\ q_i > 0,\ \operatorname{Crit}(V_{a,q}) \text{ is finite}\bigr\}.$$

Neither definition imposes genericity or nondegeneracy. Consequently, an upper bound proved only for generic or Morse configurations does not automatically bound either quantity as defined here.

1.3 Historical background and related results

In his 1873 *Treatise*, Maxwell's discussion suggested the bound $(n-1)^2$ for equilibria of n point charges [8]. The argument given there is incomplete; its modern formulation asks whether $(n-1)^2$ bounds the number of critical points when all of them are nondegenerate. A detailed historical and mathematical discussion appears in [4]. The upper-bound problem was later posed independently in the critical-point theory literature by Morse and Cairns [9].

The first bound depending only on the number of charges was obtained by Gabrielov, Novikov, and Shapiro. Using fewnomial methods, they proved that a configuration all of whose critical points are nondegenerate has at most $4^{n^2}(3n)^{2n}$ of them; for three charges they obtained the much sharper bound 12 [4]. Tsai proved the bound 4 for three charges of equal magnitude [10]. Zolotov subsequently used the Thom–Milnor theorem to bound the number of isolated critical points by $5 \cdot 9^{n+3}$ [11]. Edelsbrunner, Fillmore, and Oliveira proved that, when $n-1$ sites and all nonzero strengths are fixed, a generic position of the remaining site gives at most $2^n(3n-2)^3$ isolated critical points [3]. For the three-positive-charge problem, a July

2026 preprint of Gabrielov, Dm. Novikov, T. Novikov, and Shapiro improves the nondegenerate upper bound from 12 to 6 [5].

Lower bounds have developed in parallel. For positive charges with only nondegenerate equilibria, the classical Morse-theoretic argument gives at least $n-1$ equilibria, and collinear configurations show that this minimum is sharp [9, 3]. At the other end of the counting problem, Edelsbrunner, Fillmore, and Oliveira constructed unit positive-charge configurations with $\frac{25}{7}(n-1)$ equilibria along the sparse sequence $n = 8^\ell$ [3]. Arathoon, Ball, and Kvalheim then found five positive charges with at least 24 nondegenerate equilibria, disproving the modern nondegenerate form of Maxwell's proposed bound. Their iterative construction gives at least $10n-26$ nondegenerate equilibria for every odd $n \geq 3$ [1]. The present paper instead allows both signs. Theorem 1.2 shows that both signs are unavoidable in the symmetric moment realization used here when $N \geq 6$; it gives no obstruction to a different positive-charge construction.

1.4 Main results and proof strategy

Our main result is a quadratic lower bound for charges of arbitrary sign.

Theorem 1.1 (Main theorem)**.** *Let $N \geq 4$ be even. There is a configuration of $n = 2N$ distinct sites with nonzero charge strengths whose Coulomb potential is Morse on the complement of the sites and has a finite critical set of cardinality at least*

$$N(N-2) + 1 = (N-1)^2.$$

Consequently

$$M(n) \geq \left(\frac{n}{2} - 1\right)^2 = \frac{n^2}{4} - n + 1 \qquad (n \geq 8,\ n \equiv 0 \pmod 4).$$

Moreover, augmentation by sufficiently small charges gives

$$M(n) \geq \left(2\left\lfloor\frac{n}{4}\right\rfloor - 1\right)^2 \qquad (n \geq 8).$$

In particular, $M(n) = \Omega(n^2)$ as $n \to \infty$ through all integers.

Before the final Morse perturbation, all the critical points used in the count lie in the equatorial plane and within distance $O(\varepsilon)$ of the origin; here $\varepsilon > 0$ is the scaling parameter of the construction, introduced in Section 3. The perturbation can be taken arbitrarily small, so the final critical points remain arbitrarily close to those planar points. The perturbation is needed only to control possible additional, degenerate critical points; it does not supply any of the points used in the lower bound, which persist from the symmetric potential.

The construction does not give a corresponding quadratic lower bound for $M_+(n)$.

Theorem 1.2 (Sign obstruction)**.** *Let $N \geq 6$ be even, put $p = N/2$, and fix distinct numbers $0 < t_1 < \cdots < t_p$ and a radius $R > 0$, defining the symmetric sites of Section 3. Fix coefficients $T = (c_2, c_4, \ldots, c_{N-2}, \alpha_0, \beta_0)$ defining the local polynomial (8), with $\beta_0 \neq 0$. For the unique symmetric realizing family $\lambda_T(\varepsilon) = (q_1(\varepsilon), \ldots, q_p(\varepsilon), w(\varepsilon))$ determined by (9), there is an $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there are indices $j_+, j_- \in \{1, \ldots, p\}$ for which*

$$q_{j_+}(\varepsilon) > 0 > q_{j_-}(\varepsilon).$$

For each fixed ε in this range, the associated $2N$-component physical charge vector has a neighborhood in which every vector retains both signs among its axial-site charges. In particular, if T is the generic target fixed in (18), then ε_0 may be decreased so that, for every $0 < \varepsilon < \varepsilon_0$, the final Morse perturbation may be chosen in this neighborhood.

Our final result limits the number of critical circles available in the axisymmetric polynomial that precedes the discrete-symmetry splitting. Here *axisymmetric* means invariant under all rotations about the z-axis.

Theorem 1.3 (Critical-circle formula)**.** *Let A be a nonconstant axisymmetric harmonic polynomial of degree m on $\mathbb{R}^3$. Its critical locus is a finite union of points on the symmetry axis and circles about that axis. Write $A(x_1, x_2, z) = \widehat{A}(\rho, z)$ and let $a(s, z) = A(s, 0, z) = \widehat{A}(|s|, z)$ be its polynomial meridional restriction. For a critical circle*

$$C = \{(x_1, x_2, z_C) : x_1^2 + x_2^2 = \rho_C^2\}, \qquad \rho_C > 0,$$

let $k_C \geq 2$ be the least degree of a nonzero homogeneous term in the Taylor expansion of $a(s, z) - a(\rho_C, z_C)$ at (ρ_C, z_C). For an axial critical point $p = (0, 0, z_p)$, let $d_p \geq 2$ be the least degree of a nonzero homogeneous term in the Taylor expansion of $a(s, z) - a(0, z_p)$ at $(0, z_p)$ and put $\mu_p = d_p - 1$. These degrees are finite because A is nonconstant. Then

$$2 \sum_C (k_C - 1) + \sum_p \mu_p = m - 1, \tag{3}$$

where the sums range over all critical circles and all axial critical points, respectively, and an empty sum is understood to be zero. Consequently A has at most $\lfloor (m-1)/2 \rfloor$ critical circles. This bound is attained for every $m \geq 1$.

The proof of Theorem 1.1 starts with $N/2$ symmetric pairs of axial charges and a regular N-gon of equatorial charges. One first chooses an axisymmetric harmonic polynomial with $(N-2)/2$ nondegenerate critical circles. Adding a small sectoral harmonic $\rho^N \cos(N\phi)$ splits every circle into $2N$ nondegenerate critical points, while the origin contributes one further point. A finite moment interpolation then chooses the charge strengths so that, after rescaling near the origin, the Coulomb potentials converge in C^2 to this non-axisymmetric local polynomial. Persistence transfers all the displayed critical points to the Coulomb potential. Finally, a small charge-strength perturbation makes the potential Morse without destroying those points. The perturbation is chosen with nonzero total charge; estimates near the sites and at infinity then confine the critical set to a compact subset, so its discreteness implies finiteness. Theorem 1.2 identifies the unavoidable sign change in this realization, and Theorem 1.3 shows that the bound on the number of critical circles in the axisymmetric polynomial stage is sharp.

1.5 Organization of the paper

Section 2 fixes the harmonic notation used in the multipole expansion. Sections 3–5 construct the symmetric family, analyze its local critical points, and pass from the rescaled model to a finite Coulomb critical set. Section 6 proves the sign obstruction. Section 7 establishes the critical-circle formula for axisymmetric harmonic polynomials, and Section **??** records the consequences and limitations of the method.

2 Solid harmonics and the multipole expansion

Write $x = (x_1, x_2, z)$ and introduce cylindrical coordinates by

$$\rho = (x_1^2 + x_2^2)^{1/2}, \qquad x_1 + ix_2 = \rho e^{i\phi} \quad (\rho > 0), \qquad r = |x| = (\rho^2 + z^2)^{1/2}.$$

The angle ϕ is left undefined on the z-axis. For a nonnegative integer k, a compact set K, and a function u of class C^k on K, we write

$$\|u\| * C^k(K) = \max *0 \leq j \leq k \ \sup_K \|D^j u\|,$$

where $\|D^j u\|$ denotes the norm of the jth derivative as a symmetric j-linear form; thus $\|D^0 u\| = |u|$, $\|D^1 u\| = |\nabla u|$, and $\|D^2 u\|$ is the operator norm of the Hessian. For each integer $l \geq 0$, let P_l be the Legendre polynomial of degree l, normalized by $P_l(1) = 1$. For $x \neq 0$, set

$$Z_l(x) = r^l P_l(z/r), \tag{4}$$

and, for every $x \in \mathbb{R}^3$, set

$$W_l(x) = \mathrm{Re}\big((x_1 + ix_2)^l\big). \tag{5}$$

We use Z_l also for its polynomial extension to $\mathbb{R}^3$, whose existence and harmonicity are verified below. In particular, $Z_0 = W_0 = 1$, while for $l \geq 1$ both vanish at the origin; where $\rho > 0$, one has $W_l = \rho^l \cos(l\phi)$.

Lemma 2.1. *The functions in (4) and (5) are real homogeneous harmonic polynomials. Moreover, for every integer $j \geq 0$,*

$$P_{2j}(0) = (-1)^j \binom{2j}{j} 4^{-j}, \qquad P_{2j+1}(0) = 0. \tag{6}$$

Proof. Every monomial of $P_l(t)$ has the same parity as l. Hence $r^l P_l(z/r)$ is a polynomial in z and $r^2 = \rho^2 + z^2$, homogeneous of degree l; this gives its extension across the origin. With $t = \cos\theta$, the spherical-coordinate formula for the Laplacian and the Legendre equation

$$(1 - t^2) P_l''(t) - 2t P_l'(t) + l(l+1) P_l(t) = 0$$

give

$$\Delta\big(r^l P_l(\cos\theta)\big) = r^{l-2}\big(l(l+1) P_l(t) + (1 - t^2) P_l''(t) - 2t P_l'(t)\big) = 0$$

away from the origin, and therefore everywhere because the left-hand side is a polynomial. The polynomial W_l is homogeneous and is harmonic because it is the real part of the holomorphic polynomial $(x_1 + ix_2)^l$ and is independent of z.

Finally, set $t = 0$ in the generating function

$$(1 - 2tu + u^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(t) u^l, \qquad |u| < 1.$$

Comparison with

$$(1 + u^2)^{-1/2} = \sum_{j=0}^{\infty} (-1)^j \binom{2j}{j} 4^{-j} u^{2j}$$

gives (6). □

Lemma 2.2 (Zonal uniqueness)**.** *For each $l \geq 0$, Z_l is the unique axisymmetric homogeneous harmonic polynomial H of degree l satisfying $H(0,0,z) = z^l$. Consequently every axisymmetric harmonic polynomial A has a unique expansion*

$$A = \sum_{l=0}^{\deg A} a_l Z_l.$$

Proof. An axisymmetric homogeneous polynomial of degree l has the form

$$H(\rho, z) = \sum_{j=0}^{\lfloor l/2 \rfloor} c_j \rho^{2j} z^{l-2j}.$$

Since $\Delta(\rho^{2j}) = 4j^2 \rho^{2j-2}$, the equation $\Delta H = 0$ gives, for $0 \leq j < \lfloor l/2 \rfloor$,

$$4(j+1)^2 c_{j+1} + (l-2j)(l-2j-1)c_j = 0.$$

Thus all coefficients are determined by c_0. The polynomial Z_l is axisymmetric and harmonic by Lemma 2.1, and $Z_l(0,0,z) = z^l$ because $P_l(1) = 1$ and parity gives the same identity for $z < 0$. This proves uniqueness. Decomposing an arbitrary axisymmetric harmonic polynomial into homogeneous terms proves the final assertion. □

For a nonzero vector $a \in \mathbb{R}^3$, define the degree-l solid harmonic with axis a by

$$H_{l,a}(x) = |x|^l P_l\left(\frac{x \cdot a}{|x||a|}\right) \qquad (x \neq 0),$$

and extend it polynomially to $x = 0$ as in Lemma 2.1; thus $H_{0,a} = 1$ and $H_{l,a}(0) = 0$ for $l \geq 1$. If $\mathcal{R}$ is a rotation satisfying $\mathcal{R}(a/|a|) = (0,0,1)$, then $H_{l,a}(x) = Z_l(\mathcal{R}x)$. Thus $H_{l,a}$ is a homogeneous harmonic polynomial.

Lemma 2.3 (Multipole expansion)**.** *If a charge $q \in \mathbb{R}$ is situated at $a \neq 0$, then*

$$\frac{q}{|x-a|} = \sum_{l=0}^{\infty} \frac{q}{|a|^{l+1}} H_{l,a}(x), \qquad |x| < |a|. \tag{7}$$

For every $0 < s < |a|$ and every nonnegative integer k, the series converges in $C^k(\overline{B_s(0)})$. Its degree-l summand is a homogeneous harmonic polynomial in x.

Proof. For $0 < |x| < |a|$, apply the generating function above with

$$u = \frac{|x|}{|a|}, \qquad t = \frac{x \cdot a}{|x||a|}.$$

Since

$$|x - a| = |a|\big(1 - 2tu + u^2\big)^{1/2},$$

this gives (7). The identity at $x = 0$ follows from the polynomial extensions. The classical estimate $|P_l(t)| \leq 1$ for $-1 \leq t \leq 1$ shows absolute and uniform convergence on every closed ball $\overline{B_s(0)}$ with $s < |a|$. To obtain convergence of derivatives, choose $s < s' < |a|$ and let E_M be the difference between the left-hand side of (7) and its degree-M partial sum. The function E_M is harmonic on $B_{|a|}(0)$, and the interior derivative estimates for harmonic functions give

$$\|E_M\| * C^k(\overline{B_s(0)}) \leq C * k, s, s' \|E_M\| * C^0(\overline{B * s'(0)}).$$

The right-hand side tends to zero by uniform convergence. □

3 The symmetric family and the local limit

3.1 The sites and their allowed harmonic moments

Fix an even integer $N \geq 4$ and put

$$p = \frac{N}{2}, \qquad m = \frac{N-2}{2} = p - 1.$$

Choose distinct numbers $0 < t_1 < \cdots < t_p$ and a radius $R > 0$. The sites consist of

- the p axial pairs $(0, 0, \pm t_j)$, the two sites in the jth pair carrying the same charge q_j; and
- the vertices
$$b_u = R\left(\cos\frac{2\pi u}{N}, \sin\frac{2\pi u}{N}, 0\right), \qquad u = 0, \dots, N-1,$$
of a regular N-gon, each carrying charge w.

At this interpolation stage the parameters $q_1, \dots, q_p, w$ are arbitrary real numbers and may vanish; Section 5 will ensure that all physical charges are nonzero. There are $2p + N = 2N$ sites. Put

$$d_0 = \min(t_1, R),$$

the distance from the origin to the nearest site. By Lemma 2.3, the resulting potential has, on $B_{d_0}(0)$, a Taylor expansion

$$V(x) = \sum_{l=0}^{\infty} h_l(x),$$

where h_l is a homogeneous harmonic polynomial of degree l.

Lemma 3.1 (Symmetry restriction)**.** *All odd h_l vanish. If $l < N$, then h_l is a multiple of Z_l; at degree N there are constants α, β such that*

$$h_N = \alpha Z_N + \beta W_N.$$

Proof. Because N is even, the configuration is invariant under $x \mapsto -x$, so all odd Taylor terms vanish. An axial source contributes only a zonal harmonic. The restriction to the unit sphere of a homogeneous harmonic polynomial of degree l decomposes into azimuthal Fourier modes of integer orders k with $|k| \leq l$. For the regular N-gon, write $\theta_u = 2\pi u/N$ and let U_θ be the rotation through θ about the z-axis. Since $b_u = U_{\theta_u} b_0$, one has $H_{l,b_u} = H_{l,b_0} \circ U_{-\theta_u}$, which multiplies the mode of order k by $e^{-ik\theta_u}$. Summing over the vertices therefore produces the factor

$$\sum_{u=0}^{N-1} e^{-2\pi i k u/N},$$

which vanishes unless N divides k. Thus, when $l < N$, only $k = 0$ remains; by Lemma 2.2, this mode is a multiple of Z_l. When $l = N$, the only additional orders are $k = \pm N$, whose real span is $\mathrm{Re}\big((x_1 + ix_2)^N\big)$ and $\mathrm{Im}\big((x_1 + ix_2)^N\big)$; reflection symmetry under $x_2 \mapsto -x_2$ eliminates the imaginary mode, leaving W_N. $\square$

Lemma 3.2 (Moment isomorphism)**.** *For fixed distinct $t_1, \ldots, t_p > 0$ and $R > 0$, the linear moment map*

$$\mathcal{M} : \mathbb{R}^{p+1} \longrightarrow \mathbb{R}^{p+1}, \qquad (q_1, \ldots, q_p, w) \longmapsto (c_2, c_4, \ldots, c_{N-2}, \alpha, \beta),$$

where $h_l = c_l Z_l$ for $l = 2, 4, \ldots, N-2$ and $h_N = \alpha Z_N + \beta W_N$, is an isomorphism of $\mathbb{R}^{p+1}$.

Proof. By (7), the axial sites $(0, 0, \pm t_j)$ contribute to degree l

$$q_j t_j^{-(l+1)} \big(1 + (-1)^l\big) Z_l.$$

For a polygonal site b_u, let $(\mathcal{A}H)(x) = (2\pi)^{-1} \int_0^{2\pi} H(U_\theta x)\, d\theta$, with U_θ as in the proof of Lemma 3.1. The operator $\mathcal{A}$ extracts the axisymmetric component. Moreover,

$$H_{l,b_u}(0, 0, z) = |z|^l P_l(0) = z^l P_l(0),$$

where the last equality follows from (6). Hence $\mathcal{A}H_{l,b_u}$ is an axisymmetric homogeneous harmonic polynomial with axial restriction $P_l(0) z^l$, and Lemma 2.2 gives $\mathcal{A}H_{l,b_u} = P_l(0) Z_l$. Applying $\mathcal{A}$ to the sum over the N polygonal sites therefore gives $N P_l(0) Z_l$. Thus, for every even $l = 2, 4, \ldots, N$, the zonal coefficient has row

$$\left[2t_1^{-(l+1)}, \ldots, 2t_p^{-(l+1)} \;\middle|\; N P_l(0) R^{-(l+1)} \right].$$

Only the polygon contributes to the sectoral coefficient, whose row is

$$\left[0, \ldots, 0 \;\middle|\; 2N \binom{2N}{N} 4^{-N} R^{-(N+1)} \right].$$

Indeed, the only part of H_{N,b_u} that can contain an azimuthal mode of order N is

$$2^{-N} \binom{2N}{N} \rho^N \cos^N(\phi - \theta_u),$$

which comes from the leading term of P_N. The coefficient of $\cos\big(N(\phi - \theta_u)\big)$ in $\cos^N(\phi - \theta_u)$ is 2^{1-N}; all lower powers in P_N have azimuthal order at most $N - 2$. Since $N\theta_u = 2\pi u$, summing over the polygon contributes a factor N. Including the multipole factor $R^{-(N+1)}$ gives exactly the displayed sectoral row.

In this row the last entry is nonzero and all the others vanish. Expanding the determinant of $\mathcal{M}$ along it therefore leaves that entry times the determinant of the $p \times p$ matrix with entries $2t_j^{-(2i+1)}$, $1 \le i, j \le p$. With $s_j = t_j^{-1}$ the latter factors as

$$\big[(s_j^2)^{i-1}\big]_{i,j=1}^p \operatorname{diag}(2s_1^3, \ldots, 2s_p^3).$$

The first factor is a Vandermonde matrix in the distinct positive numbers s_j^2, and the second factor is invertible. Thus the determinant of $\mathcal{M}$ is nonzero, proving the claim. ∎

3.2 Choice of the local polynomial

Choose real coefficients $c_2, c_4, \ldots, c_{N-2}, \alpha_0, \beta_0$, with $\beta_0 \neq 0$, and set

$$P(Y) = \sum_{i=1}^{m} c_{2i} Z_{2i}(Y) + \alpha_0 Z_N(Y) + \beta_0 W_N(Y). \tag{8}$$

This is a harmonic polynomial invariant under both $Y \mapsto -Y$ and $z \mapsto -z$. For each $\varepsilon > 0$, Lemma 3.2 supplies a unique parameter vector

$$\lambda(\varepsilon) = \big(q_1(\varepsilon), \ldots, q_p(\varepsilon), w(\varepsilon)\big)$$

such that, if V_ε is the corresponding Coulomb potential and $h_l^{(\varepsilon)}$ is its degree-l Taylor term at the origin, then

$$h_{2i}^{(\varepsilon)} = \varepsilon^{-2i} c_{2i} Z_{2i} \quad (1 \le i \le m), \qquad h_N^{(\varepsilon)} = \varepsilon^{-N}(\alpha_0 Z_N + \beta_0 W_N). \tag{9}$$

Define

$$E_\varepsilon(Y) = V_\varepsilon(\varepsilon Y) - V_\varepsilon(0) - P(Y).$$

For fixed ε, this remainder is considered on $|Y| < d_0/\varepsilon$. Here and below, $B_s = B_s(0)$.

Lemma 3.3 (C^2 convergence)**.** *For every fixed $s > 0$, the function E_ε is defined on $\overline{B_s}$ for all sufficiently small ε, and*

$$\|E_\varepsilon\|_{C^2(\overline{B_s})} = O(\varepsilon^2) \qquad (\varepsilon \to 0).$$

The estimate is uniform when the target coefficients in (8) range over a fixed compact set.

Proof. By Lemma 3.2, the independent parameter vector is

$$\lambda(\varepsilon) = \varepsilon^{-N} \mathcal{M}^{-1}\big(\varepsilon^{N-2} c_2, \varepsilon^{N-4} c_4, \ldots, \varepsilon^2 c_{N-2}, \alpha_0, \beta_0\big),$$

where $\mathcal{M}$ is independent of ε. Thus, if

$$Q_1(\varepsilon) = 2\sum_{j=1}^{p} |q_j(\varepsilon)| + N|w(\varepsilon)|$$

is the total absolute charge, then $\varepsilon^N Q_1(\varepsilon)$ is bounded. This bound is uniform when the target coefficients range over a fixed compact set.

For $|Y| < d_0/\varepsilon$, homogeneity of the Taylor terms gives

$$V_\varepsilon(\varepsilon Y) - V_\varepsilon(0) = \sum_{l \ge 1} \varepsilon^l h_l^{(\varepsilon)}(Y).$$

Equations (9) make the sum through degree N equal to $P(Y)$. All odd terms vanish by Lemma 3.1; since N is even, the degree-$(N+1)$ term also vanishes. Consequently

$$E_\varepsilon(Y) = \sum_{l \ge N+2} \varepsilon^l h_l^{(\varepsilon)}(Y).$$

Since $|P_l(t)| \le 1$ for $-1 \le t \le 1$, the multipole expansion (7) gives, whenever $2\varepsilon s < d_0$,

$$\sup_{|Y|\le 2s} |E_\varepsilon(Y)| \le \frac{Q_1(\varepsilon)}{d_0} \frac{(2\varepsilon s/d_0)^{N+2}}{1-2\varepsilon s/d_0} = O(\varepsilon^2).$$

The poles of $Y \mapsto V_\varepsilon(\varepsilon Y)$ are the points a_i/ε, all at distance at least d_0/ε from the origin. Thus E_ε is harmonic on B_{2s} for sufficiently small ε. Standard interior derivative estimates for harmonic functions now give

$$\|E_\varepsilon\| * C^2(\overline{B_s}) \le C_s \|E * \varepsilon\| * C^0(\overline{B * 2s}) = O(\varepsilon^2),$$

with the same compact-uniformity in the target coefficients. □

We shall use the standard stability of nondegenerate zeros in the following quantitative form, which applies to arbitrary families and not only to sequences.

Lemma 3.4 (Persistence)**.** *Let f be C^2 on a neighborhood U of a nondegenerate critical point Y_0. Then, for every sufficiently small closed ball $\overline{B} \subset U$ centered at Y_0, there is an $\eta > 0$ with the following property: every $g \in C^2(\overline{B})$ with $\|g - f\|_{C^2(\overline{B})} < \eta$ has exactly one critical point in B, and that point is nondegenerate. Since $\overline{B}$ may be taken arbitrarily small, this locates the critical point of g arbitrarily close to Y_0.*

Proof. Put $H = \operatorname{Hess} f(Y_0)$, which is invertible. Because $\operatorname{Hess} f$ is continuous, every sufficiently small closed ball $\overline{B} \subset U$ centered at Y_0 satisfies

$$\sup_{\overline{B}} \|H^{-1}(\operatorname{Hess} f - H)\| < \frac{1}{4}; \tag{10}$$

fix such a ball. If $x, y \in \overline{B}$, then integrating $\operatorname{Hess} f$ along the line segment from y to x and using (10) gives

$$\left|H^{-1}(\nabla f(x) - \nabla f(y)) - (x-y)\right| \le \frac{1}{4}|x-y|. \tag{11}$$

Taking $y = Y_0$ yields $|H^{-1}\nabla f(x)| \ge \frac{3}{4}|x - Y_0|$, so Y_0 is the only critical point of f in $\overline{B}$ and

$$\mu := \min_{\partial B} |\nabla f| > 0.$$

Moreover, for $0 \le \tau \le 1$ and $x \in \partial B$, the same estimate gives

$$\left|H^{-1}\big(\tau \nabla f(x) + (1-\tau)H(x - Y_0)\big)\right| \ge |x - Y_0| - \frac{\tau}{4}|x - Y_0| \ge \frac{3}{4}|x - Y_0| > 0,$$

so this homotopy from ∇f to the linear map $x \mapsto H(x - Y_0)$ does not vanish on ∂B, and therefore

$$\deg(\nabla f, B, 0) = \operatorname{sgn}\det H \ne 0.$$

Now set $\eta = \min\{(4\|H^{-1}\|)^{-1}, \mu\} > 0$ and let $g \in C^2(\overline{B})$ satisfy $\|g - f\|_{C^2(\overline{B})} < \eta$. Then

$$\sup_{\overline{B}} \|H^{-1}(\operatorname{Hess} g - H)\| \le \|H^{-1}\| \sup_{\overline{B}} \|\operatorname{Hess} g - \operatorname{Hess} f\| + \sup_{\overline{B}} \|H^{-1}(\operatorname{Hess} f - H)\| < \frac{1}{2},$$

and $\sup_{\partial B} |\nabla g - \nabla f| < \mu$, so the straight-line homotopy between ∇f and ∇g is nonzero on ∂B. Hence $\deg(\nabla g, B, 0) = \operatorname{sgn}\det H \ne 0$ and ∇g has a zero in B. Repeating (11) for g, with $\frac{1}{2}$ in place of $\frac{1}{4}$, gives

$$\left|H^{-1}(\nabla g(x) - \nabla g(y)) - (x-y)\right| \le \frac{1}{2}|x-y| \qquad (x, y \in \overline{B}),$$

so ∇g is injective on $\overline{B}$ and its zero in B is unique. Finally, $\|H^{-1}\operatorname{Hess} g - I\| < \frac{1}{2}$ on $\overline{B}$ forces $H^{-1}\operatorname{Hess} g$, and hence $\operatorname{Hess} g$, to be invertible there; the zero is therefore nondegenerate. □

4 Critical points of the local polynomial

Write
$$P(\rho, \phi, z) = A(\rho, z) + \beta_0 \rho^N \cos(N\phi),$$
where A is axisymmetric and even in z.

Lemma 4.1. *Every off-axis critical point of P lies in one of the $2N$ vertical half-planes*
$$\phi = \frac{j\pi}{N}, \qquad j = 0, \dots, 2N-1,$$
where the azimuthal angle is understood modulo 2π. In a half-plane of parity $\sigma = (-1)^j \in \{+1, -1\}$, its remaining critical-point equations are
$$A_\rho + \sigma N \beta_0 \rho^{N-1} = 0, \qquad A_z = 0. \tag{12}$$
For each fixed parity, exactly N of the displayed half-planes give the same pair of equations.

Proof. Because $\beta_0 \neq 0$, for $\rho > 0$ the azimuthal critical-point equation is
$$P_\phi = -N\beta_0 \rho^N \sin(N\phi) = 0,$$
and hence $\sin(N\phi) = 0$. Thus $\phi = j\pi/N$ modulo 2π, and substitution gives (12). Among $j = 0, \dots, 2N-1$, exactly N indices have either parity. □

On $z = 0$, the second equation in (12) holds identically. Put $u = \rho^2$. The first equation is $\rho\Phi_\sigma(u) = 0$, where
$$\Phi_\sigma(u) = \sum_{i=1}^{m} 2i\, c_{2i} P_{2i}(0) u^{i-1} + N\big(\alpha_0 P_N(0) + \sigma\beta_0\big) u^m. \tag{13}$$
Indeed, $Z_{2i}(\rho, 0) = P_{2i}(0)\rho^{2i}$ and $Z_N(\rho, 0) = P_N(0)\rho^N$, while $N = 2m + 2$; differentiating the restriction of P to a ray of parity σ therefore gives $P_\rho = \rho\Phi_\sigma(\rho^2)$. Notice that $P_N(0) \neq 0$ because N is even.

Proposition 4.2 (Design of the roots)**.** *The coefficients in (8) may be chosen so that each of Φ_+ and Φ_- has m distinct positive roots. Every resulting equatorial critical point, as well as the origin, is nondegenerate. Hence P has at least*
$$N(2m) + 1 = N(N-2) + 1 = (N-1)^2$$
distinct nondegenerate critical points.

Proof. Choose distinct numbers $0 < u_1 < \dots < u_m$ and a nonzero α_0. Put
$$\beta_0 = \delta\alpha_0 P_N(0),$$
where $0 < |\delta| < 1$ will be chosen sufficiently small. Prescribe
$$\Phi_+(u; \delta) = N\big(\alpha_0 P_N(0) + \beta_0\big) \prod_{r=1}^{m} (u - u_r). \tag{14}$$

Here and below, the argument after the semicolon records the dependence of the coefficients on δ. Matching the coefficients below the leading term in (13) determines $c_2, \dots, c_{N-2}$ uniquely, because $2iP_{2i}(0) \neq 0$ by (6). These coefficients depend continuously on δ (in fact, affinely). Moreover,

$$\Phi_-(u;\delta) = \Phi_+(u;\delta) - 2N\beta_0 u^m. \tag{15}$$

The leading coefficients of $\Phi_+(\,\cdot\,;\delta)$ and $\Phi_-(\,\cdot\,;\delta)$ are, respectively,

$$N\alpha_0 P_N(0)(1+\delta) \quad \text{and} \quad N\alpha_0 P_N(0)(1-\delta),$$

so both polynomials have degree m. Define

$$\Phi_0(u) := N\alpha_0 P_N(0) \prod_{r=1}^{m} (u - u_r).$$

As $\delta \to 0$, both $\Phi_+(\,\cdot\,;\delta)$ and $\Phi_-(\,\cdot\,;\delta)$ converge coefficientwise to Φ_0. For each $\sigma \in \{+1,-1\}$, apply the real implicit function theorem to $\Phi_\sigma(u;\delta) = 0$ at $(u,\delta) = (u_r, 0)$. After intersecting the resulting parameter neighborhoods and shrinking the root neighborhoods, we obtain pairwise disjoint open intervals $I_r \Subset (0,\infty)$ and, for all sufficiently small $|\delta|$, unique simple zeros

$$u_{r,\sigma}(\delta) \in I_r, \qquad u_{r,\sigma}(\delta) \longrightarrow u_r \quad (\delta \to 0).$$

Thus each $\Phi_\sigma(\,\cdot\,;\delta)$ has at least m distinct positive roots. Since its degree is m, these are all of its roots.

We now check nondegeneracy. Fix r and σ, set $u = u_{r,\sigma}(\delta)$ and $\rho = \sqrt{u}$, and choose any half-plane $\phi = j\pi/N$ with $(-1)^j = \sigma$. Evenness in z gives $P_{\rho z} = P_{\phi z} = 0$ at $z = 0$, while $\sin(N\phi) = 0$ gives $P_{\rho\phi} = 0$. In the orthonormal cylindrical frame (e_ρ, e_ϕ, e_z) the off-diagonal entries of the Euclidean Hessian are

$$\operatorname{Hess} P(e_\rho, e_\phi) = \rho^{-1}P_{\rho\phi} - \rho^{-2}P_\phi, \qquad \operatorname{Hess} P(e_\rho, e_z) = P_{\rho z}, \qquad \operatorname{Hess} P(e_\phi, e_z) = \rho^{-1}P_{\phi z},$$

and all three vanish at the point in question, the first because $P_\phi = 0$ there as well. Thus the Hessian is diagonal in that frame. At a critical point its azimuthal entry is

$$\operatorname{Hess} P(e_\phi, e_\phi) = \rho^{-2}P_{\phi\phi} + \rho^{-1}P_\rho = -N^2\beta_0\sigma\rho^{N-2},$$

where we used $P_\rho = 0$ and $\cos(N\phi) = \sigma$. Differentiating $P_\rho = \rho\Phi_\sigma(\rho^2;\delta)$ gives the radial entry

$$P_{\rho\rho} = \Phi_\sigma(u;\delta) + 2u\,\partial_u\Phi_\sigma(u;\delta) = 2u\,\partial_u\Phi_\sigma(u;\delta).$$

Consequently, the three eigenvalues are

$$\lambda_\phi = -N^2\beta_0\sigma u^{(N-2)/2}, \qquad \lambda_\rho = 2u\,\partial_u\Phi_\sigma(u;\delta), \qquad \lambda_z = -(\lambda_\phi + \lambda_\rho). \tag{16}$$

The last identity follows because the trace of the Euclidean Hessian is $\Delta P = 0$. The first two eigenvalues are nonzero because $\beta_0 \neq 0$, $u > 0$, and the roots are simple. Along the root branch $u_{r,\sigma}(\delta)$, as $\delta \to 0$ the third tends to

$$-2u_r\Phi_0'(u_r) \neq 0.$$

There are only finitely many branches, so one sufficiently small nonzero δ makes every λ_z nonzero simultaneously.

The origin is a critical point because P has no terms of degree below two. The constant term of (14) is

$$N\alpha_0 P_N(0)(1+\delta)(-1)^m \prod_{r=1}^{m} u_r \neq 0.$$

By (13), this constant term equals $2c_2P_2(0)$, so $c_2 \neq 0$. Since $Z_2 = z^2 - \rho^2/2$, the Hessian at the origin is $\mathrm{diag}(-c_2, -c_2, 2c_2)$ and is nonsingular. Each root of Φ_σ yields one critical point in each of the N half-planes of parity σ, and the two parities use disjoint sets of half-planes. Hence there are $2Nm = N(N-2)$ off-axis equatorial points and, in addition, the origin. □

The preceding construction also shows that the set of target vectors $T = (c_2, \ldots, c_{N-2}, \alpha_0, \beta_0)$ for which the conclusion of Proposition 4.2 holds—that is, for which Φ_+ and Φ_- each have m distinct positive roots and the resulting $N(N-2)$ equatorial points and the origin are nondegenerate critical points of P—is a nonempty open subset of the coefficient space. Indeed, the coefficients of $\Phi_\pm$ depend linearly on T, and near the constructed vector their leading coefficients stay nonzero, so both polynomials keep degree m and their roots remain simple and stay within the disjoint positive intervals I_r. The inequalities $\beta_0 \neq 0$, $c_2 \neq 0$, and $\lambda_z \neq 0$ at the finitely many corresponding critical points also persist by continuity; therefore all three Hessian eigenvalues remain nonzero. We shall use this openness below.

5 Passage to a Coulomb potential and finiteness

We first choose the target coefficients and only then fix the scale ε. Let

$$\mathcal{U} \subset \mathbb{R}^{p+1}$$

be the nonempty open set of target vectors $T = (c_2, c_4, \ldots, c_{N-2}, \alpha_0, \beta_0)$ supplied by Proposition 4.2. The formula in the proof of Lemma 3.3 may be written as

$$\varepsilon^N \lambda_T(\varepsilon) = \mathcal{M}^{-1} D_\varepsilon T, \qquad D_\varepsilon = \mathrm{diag}(\varepsilon^{N-2}, \varepsilon^{N-4}, \ldots, \varepsilon^2, 1, 1). \tag{17}$$

Consequently each of $\varepsilon^N q_j(\varepsilon)$ and $\varepsilon^N w(\varepsilon)$ is a polynomial in ε^2 whose coefficients depend linearly on T. The same is true of $\varepsilon^N Q_T(\varepsilon)$, where

$$Q_T(\varepsilon) = 2\sum_{j=1}^{p} q_j(\varepsilon) + Nw(\varepsilon)$$

is the total charge.

For a linear functional ℓ on the charge-parameter space $\mathbb{R}^{p+1}$, let $\ell\big(\varepsilon^N \lambda_T(\varepsilon)\big)$ denote the associated polynomial in ε^2. Let ℓ range over the $p+1$ coordinate functionals and over the nonzero functional

$$L(q_1, \ldots, q_p, w) = 2\sum_{j=1}^{p} q_j + Nw,$$

so that these polynomials are exactly $\varepsilon^N q_1(\varepsilon), \ldots, \varepsilon^N q_p(\varepsilon), \varepsilon^N w(\varepsilon)$ and $\varepsilon^N Q_T(\varepsilon)$. For each such ℓ, the set of targets T for which the associated polynomial vanishes identically is contained in a proper linear subspace of $\mathbb{R}^{p+1}$. Indeed, evaluating (17) at $\varepsilon = 1$ gives $\lambda_T(1) = \mathcal{M}^{-1}T$. Because $\mathcal{M}^{-1}$ is invertible, the composition of any listed nonzero functional with $\mathcal{M}^{-1}$ is a nonzero functional. A finite union of proper linear subspaces cannot contain the nonempty open set $\mathcal{U}$. We therefore fix

$$T \in \mathcal{U} \tag{18}$$

outside this union. A nonzero polynomial in ε^2 has no zeros for all sufficiently small positive ε. Thus every physical charge and $Q_T(\varepsilon)$ are nonzero for all sufficiently small positive ε.

For this fixed target, set

$$F_\varepsilon(Y) := V_\varepsilon(\varepsilon Y) - V_\varepsilon(0) = P(Y) + E_\varepsilon(Y).$$

Apply Lemma 3.4 to P at each of the $(N-1)^2$ critical points constructed in Proposition 4.2, taking the balls small enough to be pairwise disjoint and to have all their closures contained in one fixed ball $\overline{B_s}$; let $B_1, \dots, B_{(N-1)^2}$ be these balls and let $\eta > 0$ be the smallest of the corresponding tolerances. By Lemma 3.3, $\|F_\varepsilon - P\|_{C^2(\overline{B_s})} = O(\varepsilon^2)$, so this quantity is smaller than η for all sufficiently small $\varepsilon > 0$. For such ε, each B_k contains exactly one critical point $Y_{\varepsilon,k}$ of F_ε, and that point is nondegenerate; the balls being disjoint, these $(N-1)^2$ points are distinct. The chain rule gives

$$\nabla_Y F_\varepsilon(Y) = \varepsilon \nabla_x V_\varepsilon(\varepsilon Y), \qquad \operatorname{Hess}_Y F_\varepsilon(Y) = \varepsilon^2 \operatorname{Hess}_x V_\varepsilon(\varepsilon Y). \tag{19}$$

Hence $x_{\varepsilon,k} = \varepsilon Y_{\varepsilon,k}$ are distinct nondegenerate critical points of V_ε and $|x_{\varepsilon,k}| = O(\varepsilon)$. Fix from now on one sufficiently small ε for which all these conclusions and all the preceding nonvanishing conditions hold.

All these points lie in the equatorial plane. Indeed, the two sites of each axial pair carry the same charge and the polygonal sites lie in $z = 0$, so V_ε, and therefore F_ε, is invariant under the reflection $z \mapsto -z$. Every critical point of P listed in Proposition 4.2 has $z = 0$, so each ball B_k is invariant under that reflection; since the reflection maps critical points of F_ε to critical points of F_ε, the uniqueness in Lemma 3.4 forces $Y_{\varepsilon,k}$ to be a fixed point of it. Hence $z = 0$ at $Y_{\varepsilon,k}$, and likewise at $x_{\varepsilon,k}$. This planarity is recorded only for orientation; it is not used below.

Lemma 5.1 (Morse perturbation)**.** *Suppose the sites $a_1, \dots, a_n$ are not contained in an affine plane. Then, for almost every charge vector $q \in \mathbb{R}^n$, the potential $V_{a,q}$ is Morse on Ω_a. In particular, such charge vectors are dense in $\mathbb{R}^n$.*

Proof. Consider

$$F : \Omega_a \times \mathbb{R}^n \longrightarrow \mathbb{R}^3, \qquad F(x,q) = \nabla_x V_{a,q}(x).$$

Its derivative in the q_i direction is

$$D_{q_i} F(x,q) = -\frac{x - a_i}{|x - a_i|^3}.$$

If the vectors $x - a_i$ did not span $\mathbb{R}^3$, there would be a nonzero vector v such that $v \cdot a_i = v \cdot x$ for every i, placing all the sites in one affine plane. Thus $D_q F$ is surjective at every (x,q), so F is a submersion and $Z := F^{-1}(0)$ is a smooth submanifold of dimension $(3+n) - 3 = n$.

Let $\pi : Z \to \mathbb{R}^n$ be the restriction of the projection $(x,q) \mapsto q$. A direct examination of the tangent-space equation

$$D_x F(x,q)\, \xi + D_q F(x,q)\, \eta = 0$$

shows that q is a regular value of π exactly when $D_x F(x,q)$ is surjective at every $(x,q) \in Z$. Since $D_x F = \operatorname{Hess} V_{a,q}$ is a square matrix, this is precisely the assertion that every critical point of $V_{a,q}$ is nondegenerate. Sard's theorem therefore shows that the exceptional charge vectors form a set of Lebesgue measure zero. This application is valid on the noncompact manifold Z, since Sard's theorem is local and Z is second countable. The complement of a null set is dense. See, for example, [6, Chapter 2]. $\square$

Our sites are not coplanar: the polygon has $N \geq 4$ vertices, which affinely span the plane $z = 0$, while every axial site lies off that plane. Regard the symmetric strengths as a vector $\mathbf{q}^{\mathrm{sym}} \in \mathbb{R}^{2N}$, with repeated entries for sites in the same orbit. Apply Lemma 3.4 to $V_\varepsilon = V_{a,\mathbf{q}^{\mathrm{sym}}}$ at each of the $(N-1)^2$ nondegenerate critical points $x_{\varepsilon,k}$, taking the balls small enough to be pairwise disjoint and to have closures in Ω_a, and let $\eta' > 0$ be the smallest of the corresponding tolerances. A potential depends linearly on its charge vector, and the sites are at a positive distance from the union $\mathcal{K}$ of those closed balls, so $\|V_{a,q'} - V_{a,q}\|_{C^2(\mathcal{K})} \leq C_{\mathcal{K}}|q' - q|$ for a constant $C_{\mathcal{K}}$ depending only on $\mathcal{K}$ and the sites. Together with the nonvanishing of every charge and of the total charge at $\mathbf{q}^{\mathrm{sym}}$, this yields an open neighborhood of $\mathbf{q}^{\mathrm{sym}}$ in $\mathbb{R}^{2N}$ in which every charge and the total charge remain nonzero and in which each of the $(N-1)^2$ pairwise disjoint balls still contains exactly one critical point, a nondegenerate one. Lemma 5.1 allows us to choose within this neighborhood a charge vector for which the resulting potential, henceforth denoted by $V_{a,q}$, is Morse. Thus every critical point of the perturbed potential, whether or not among the persisted points, is nondegenerate. Write

$$Q := \sum_{i=1}^{2N} q_i$$

for its total charge; the choice of neighborhood ensures that $Q \neq 0$.

It remains to verify that the resulting critical set is finite. Near a site a_i,

$$|\nabla V_{a,q}(x)| \geq \frac{|q_i|}{|x - a_i|^2} - \sum_{j \neq i} \frac{|q_j|}{|x - a_j|^2},$$

and the second term is bounded in a sufficiently small fixed neighborhood of a_i. Since $q_i \neq 0$, the first term tends to infinity as $x \to a_i$; therefore no critical point lies in some punctured ball $B_{r_i}(a_i) \setminus \{a_i\}$. At infinity, uniformly in $\omega = x/|x|$,

$$\nabla V_{a,q}(x) = -Q\frac{x}{|x|^3} + O(|x|^{-3}),$$

because

$$\frac{x - a_i}{|x - a_i|^3} = \frac{x}{|x|^3} + O(|x|^{-3}).$$

Thus, for some constant C and all sufficiently large $|x|$,

$$|\nabla V_{a,q}(x)| \geq \frac{|Q|}{|x|^2} - \frac{C}{|x|^3} > 0.$$

The critical set is consequently the zero set of the continuous map $\nabla V_{a,q}$ inside a compact set of the form, for a suitable R_∞,

$$\overline{B_{R_\infty}(0)} \setminus \bigcup_i B_{r_i}(a_i),$$

and is therefore compact. Since $V_{a,q}$ is Morse, its critical points are isolated. A compact discrete subset of Euclidean space is finite, so the resulting potential has a finite critical set. This completes the construction for $n = 2N$.

For completeness, the lower bound extends from multiples of four to every $n \geq 8$.

Lemma 5.2 (Augmentation)**.** *Suppose an n-charge potential is Morse, has nonzero total charge, and has a finite critical set of cardinality K, and suppose its sites are not coplanar. Then there is an $(n+1)$-charge potential with the same properties, with noncoplanar sites, and with at least K critical points.*

Proof. Write Q for the total charge of the original potential. Choose a point b outside both the old sites and the finite set $\mathrm{Crit}(V_{a,q})$, and add at b a charge of strength $t \neq 0$. The enlarged set of sites remains noncoplanar. Apply Lemma 3.4 to the original potential at each of its K critical points, taking the balls small enough to be pairwise disjoint and to have closures avoiding all old sites and b; let $\eta > 0$ be the smallest of the corresponding tolerances. On the union of those closures,

$$\left\| \frac{t}{|x-b|} \right\|_{C^2} \longrightarrow 0 \qquad (t \to 0),$$

so for all sufficiently small $|t| \neq 0$ each of the K balls contains exactly one critical point of the enlarged potential, and that point is nondegenerate. We also choose $0 < |t| < |Q|/2$, so the enlarged total charge $Q + t$ is nonzero.

Exactly as in the paragraph preceding this lemma, linearity of the potential in its charge vector and Lemma 3.4 now give an open neighborhood of the enlarged charge vector in which all $n+1$ charges and the total charge remain nonzero and each of the K balls still contains a critical point. By Lemma 5.1, this neighborhood contains a charge vector for which the enlarged potential is Morse. The compactness argument above then makes its critical set finite, and it still contains at least the K persisted points. $\square$

Given $n \geq 8$, let $n_0 = 4\lfloor n/4 \rfloor$. Then $n_0 \geq 8$, $n - n_0 \in \{0, 1, 2, 3\}$, and $N = n_0/2$ is even. The construction for n_0, followed by $n - n_0$ applications of Lemma 5.2, gives

$$M(n) \geq \left(\frac{n_0}{2} - 1\right)^2 = \left(2\left\lfloor \frac{n}{4} \right\rfloor - 1\right)^2.$$

This proves all the assertions of Theorem 1.1.

Remark 5.3 (The unperturbed angular dependence)**.** The symmetric potential itself has no horizontal critical circle centered on the z-axis. Indeed, the contribution of the axial charges to the potential is independent of ϕ. A candidate circle has fixed $\rho > 0$ and z and is contained in the domain of the potential. Hence $(\rho, z) \neq (R, 0)$, because the circle with $(\rho, z) = (R, 0)$ contains all polygonal sites. For the polygonal charge at b_u, let $\psi = \phi - 2\pi u/N$ be the difference between the azimuthal angles of the field point and of that charge. Its contribution to the potential, apart from the constant factor w, is

$$(A - B\cos\psi)^{-1/2}, \qquad A = \rho^2 + z^2 + R^2, \quad B = 2\rho R.$$

Here $A > B > 0$. Set

$$\eta = \frac{A - \sqrt{A^2 - B^2}}{B} \in (0, 1), \qquad C = \frac{B}{2\eta} > 0.$$

Then

$$A - B\cos\psi = C(1-\eta e^{i\psi})(1-\eta e^{-i\psi}).$$

If $\gamma_k = \binom{2k}{k}4^{-k}$, the binomial series shows that the complex Fourier coefficient of order $k \geq 0$, in ψ, of the product of the two inverse square roots is

$$\kappa_k = \sum_{l=0}^{\infty} \gamma_{l+k}\gamma_l \eta^{2l+k} > 0.$$

Summing over the N rotated polygonal charges retains precisely the modes divisible by N; in particular, the coefficient of $\cos(N\phi)$ is $2NwC^{-1/2}\kappa_N \neq 0$ because $w \neq 0$. On a critical circle, V_ϕ vanishes identically, so the restriction of V to that circle is constant, contradicting the nonzero Fourier coefficient. This observation does not exclude more general nonisolated critical sets; that is why the proof above uses the Morse perturbation.

6 The sign obstruction

Proof of Theorem 1.2. Fix the target T in the statement. Only the polygon contributes to the sectoral moment. By the last row in the proof of Lemma 3.2, the coefficient of W_N in the degree-N Taylor term is

$$2N\binom{2N}{N}4^{-N}R^{-(N+1)}w(\varepsilon).$$

Equating this with the prescribed coefficient $\varepsilon^{-N}\beta_0$ from (9) gives

$$\widetilde{w} := \varepsilon^N w(\varepsilon) = \frac{\beta_0 R^{N+1}4^N}{2N\binom{2N}{N}},$$

which is nonzero and independent of ε because $\beta_0 \neq 0$.

For $1 \leq i < p$, the zonal row of index $l = 2i$ in the same moment map, together with (9), gives

$$2\sum_{j=1}^{p} q_j(\varepsilon)t_j^{-(2i+1)} + Nw(\varepsilon)P_{2i}(0)R^{-(2i+1)} = \varepsilon^{-2i}c_{2i}. \tag{20}$$

Define

$$\widetilde{g}_j(\varepsilon) = 2\varepsilon^N q_j(\varepsilon)t_j^{-3}, \qquad v_j = t_j^{-2} > 0.$$

Multiplying (20) by ε^N yields

$$\sum_{j=1}^{p} \widetilde{g}_j(\varepsilon)v_j^{i-1} = \varepsilon^{N-2i}c_{2i} - \widetilde{w}\,NP_{2i}(0)R^{-(2i+1)}. \tag{21}$$

Because $i < p = N/2$, one has $N - 2i \geq 2$. Hence, using (6), the right-hand side converges as $\varepsilon \to 0$ to

$$L_i := -\widetilde{w}\,NP_{2i}(0)R^{-(2i+1)} = (-1)^{i+1}\widetilde{w}\,N\binom{2i}{i}4^{-i}R^{-(2i+1)}. \tag{22}$$

Every L_i is nonzero, and its sign alternates with i. When $N \geq 6$, we have $p - 1 \geq 2$, so the indices $i = 1, 2$ are both available; their limits are, explicitly,

$$L_1 = \frac{N\widetilde{w}}{2R^3}, \qquad L_2 = -\frac{3N\widetilde{w}}{8R^5}.$$

They are nonzero and have opposite signs. Therefore there is an $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$, the two corresponding right-hand sides of (21) are also nonzero and have opposite signs.

Fix such an ε. If the numbers $q_j(\varepsilon)$ did not include both signs, then either $q_j(\varepsilon) \geq 0$ for every j or $q_j(\varepsilon) \leq 0$ for every j. Since $2\varepsilon^N t_j^{-3} > 0$ and $v_j > 0$, all the numbers $\widetilde{g}_j(\varepsilon)$ would have the same weak sign, and the two sums $\sum_j \widetilde{g}_j(\varepsilon) v_j^{i-1}$ for $i = 1, 2$ would then both be nonnegative or both be nonpositive. This contradicts the opposite nonzero signs just obtained. Thus there are indices j_+ and j_- with $q_{j_+}(\varepsilon) > 0 > q_{j_-}(\varepsilon)$.

Regard the common orbit strengths as the physical vector

$$\mathbf{q}^{\mathrm{sym}}(\varepsilon) := \big(q_1(\varepsilon), q_1(\varepsilon), \ldots, q_p(\varepsilon), q_p(\varepsilon), \underbrace{w(\varepsilon), \ldots, w(\varepsilon)}\, {*N} \text{ entries}\big) \in \mathbb{R}^{2N},$$

where the first two coordinates in each displayed pair correspond to the sites $(0, 0, \pm t_j)$. Define

$$\eta * \operatorname{sign}(\varepsilon) := \frac{1}{2} \min\big\{q_{j_+}(\varepsilon), -q_{j_-}(\varepsilon)\big\} > 0$$

and let $\mathbf{q}' \in \mathbb{R}^{2N}$ satisfy $\|\mathbf{q}' - \mathbf{q}^{\mathrm{sym}}(\varepsilon)\|_\infty < \eta_{\mathrm{sign}}(\varepsilon)$. Then $\mathbf{q}'$ has positive charges at the two sites of the j_+ axial pair and negative charges at the two sites of the j_- axial pair. This proves the neighborhood assertion without any genericity assumption on T.

For the target fixed in (18), decrease ε_0 so that the nonvanishing and persistence conclusions of Section 5 hold whenever $0 < \varepsilon < \varepsilon_0$. For each such ε, intersect the corresponding open perturbation neighborhood from that section with the maximum-norm ball of radius $\eta_{\mathrm{sign}}(\varepsilon)$. The intersection is an open neighborhood of $\mathbf{q}^{\mathrm{sym}}(\varepsilon)$, so the density conclusion of Lemma 5.1 supplies a Morse charge vector in it. This choice preserves the counted critical points, the nonvanishing of every charge and of the total charge, and both axial signs. $\square$

The result is an obstruction for this realizing family and its sufficiently nearby perturbations, not for every possible positive-charge construction. The threshold $N \geq 6$ is where two lower zonal moment equations first become available; the proof gives no sign obstruction when $N = 4$. The argument also gives no information about whether the signs alternate when the axial pairs are ordered by t_j.

7 Axisymmetric harmonic polynomials

We now prove Theorem 1.3. Throughout this section m denotes the degree of the axisymmetric polynomial under consideration and p denotes an axial critical point; both usages are local to this section and to the statement of Theorem 1.3, and are unrelated to the integers $m = (N-2)/2$ and $p = N/2$ of Section 3. Write $A(x_1, x_2, z) = \widehat{A}(\rho, z)$, where $\rho = (x_1^2 + x_2^2)^{1/2}$, and extend its restriction to a meridian plane across the axis by

$$a(s, z) = A(s, 0, z) = \widehat{A}(|s|, z), \qquad s \in \mathbb{R}.$$

We suppress the hat when no confusion can result. The function a is a polynomial, even in s. For $\rho > 0$, axisymmetry gives

$$\nabla A = A_\rho \, e_\rho + A_z \, e_z.$$

Thus a zero (ρ, z) of (A_ρ, A_z) represents a full critical circle and corresponds to the two critical points $(\pm\rho, z)$ of a. On the axis, rotational invariance makes the two horizontal derivatives vanish, so $(0,0,z)$ is critical exactly when $\frac{d}{dz}A(0,0,z)=0$; it corresponds to the single critical point $(0,z)$ of a. We call a critical circle *meridionally nondegenerate* if either corresponding critical point of a has nonsingular planar Hessian; equivalently, the circle is a Morse–Bott critical submanifold of A.

For an isolated zero y of a planar vector field F, the *local index* of F at y is the topological degree, or winding number, of $F/|F|$ on a sufficiently small positively oriented circle about y. We use the same convention for the degree of F on a large positively oriented circle.

Lemma 7.1. *For $l \geq 1$, set $\zeta_l(s,z) = Z_l(s,0,z)$. The planar vector field $\nabla_{s,z}\zeta_l$ is nonzero away from the origin and has degree $1-l$ on every positively oriented circle centered at the origin.*

Proof. The l roots of P_l are real, simple, and lie in $(-1,1)$. It follows from $\zeta_l = r^l P_l(z/r)$ that the zero set of ζ_l consists of l distinct lines through the origin: the $2l$ zeros on the unit circle arising from the simple Legendre roots pair antipodally. Homogeneity and degree l therefore make ζ_l, up to a nonzero constant, the product of defining linear forms for these lines. Euler's identity shows that a zero of $\nabla_{s,z}\zeta_l$ away from the origin would also be a zero of ζ_l. Such a point lies on exactly one of the l lines, where the product rule makes the gradient nonzero. Hence $\nabla_{s,z}\zeta_l$ is nonzero away from the origin; for $l \geq 2$, homogeneity also shows that the origin is its unique zero, whereas for $l=1$ the gradient is constant and has no zero.

Continuously deform the l distinct lines, without allowing two of them to coincide, to l equally spaced lines; this is possible because the lines correspond to l distinct points of the circle $\mathbb{RP}^1$, and the space of l-element subsets of a circle is connected. Throughout the deformation the polynomial remains a product of l pairwise nonproportional linear forms, so by the argument just given its gradient remains nonzero on the unit circle, and its degree is unchanged. The product of the equally spaced linear forms is, after a rotation and nonzero rescaling, $\operatorname{Re}\big((s+iz)^l\big)$. Identifying a planar vector with a complex number, the gradient of this polynomial is $l(s-iz)^{l-1}$ up to a fixed nonzero complex factor. Its degree is therefore $-(l-1)=1-l$. ☐

Proof of Theorem 1.3. First we justify finiteness of the critical locus modulo rotations. On the axis, the axial critical points are precisely the zeros of the polynomial $\frac{d}{dz}A(0,0,z)$. If this derivative vanished identically, then $A(0,0,z)$ would be constant; the unique zonal expansion from Lemma 2.2 would then make A constant. Hence a nonconstant A has only finitely many axial critical points.

In the open half-plane $\rho>0$, the function $A(\rho,z)$ satisfies the analytic elliptic equation

$$A_{\rho\rho} + A_{zz} + \rho^{-1}A_\rho = 0. \tag{23}$$

Consider the quotient critical set

$$S = \{(\rho,z) : \rho>0,\ A_\rho(\rho,z) = A_z(\rho,z) = 0\}.$$

It is semialgebraic because $A(\rho,z)$ is a polynomial in ρ^2 and z. If S were infinite, its finite stratification into connected Nash submanifolds would contain a positive-dimensional stratum and therefore a regular real-analytic arc Γ; see [2, Proposition 2.9.10]. The restriction of A to Γ is constant, say equal to c, because its tangential derivative vanishes. Its normal derivative also vanishes there since $\nabla_{\rho,z}A = 0$ on S. Thus $u = A-c$ has zero Cauchy data on a regular subarc of Γ. Choose one side of that subarc and define $\widetilde{u}$ to equal u there and zero on the other side. The vanishing of both u and its normal derivative makes $\widetilde{u}$ a distributional solution of (23) in a neighborhood of the subarc. The coefficients of the operator are analytic for $\rho>0$, and its principal symbol $\xi_\rho^2 + \xi_z^2$ makes every regular arc noncharacteristic. Holmgren's uniqueness theorem therefore makes $\widetilde{u}$ vanish near the subarc; see [7, Theorem 8.6.5]. Hence the original u vanishes on a nonempty open set. Analytic continuation on the connected half-plane, followed by the polynomial identity principle, would make A constant on $\mathbb{R}^3$, a contradiction. Hence S is finite.

For later use, we also record explicitly that no critical point lies sufficiently far from the origin. By Lemma 2.2, $A = \sum_{l=0}^{m} a_l Z_l$ with $a_m \neq 0$, so the degree-m homogeneous term of a is $a_m \zeta_m$. Uniformly for $\theta \in S^1$,

$$r^{1-m} \nabla_{s,z} a(r\theta) = a_m \nabla_{s,z} \zeta_m(\theta) + O(r^{-1}) \qquad (r \to \infty). \tag{24}$$

By Lemma 7.1, the limiting vector field has a positive minimum norm on S^1. Thus the left-hand side is nonzero for all sufficiently large r. Together with the preceding arguments, this proves that the critical locus is a finite union of axial points and off-axis circles.

Let (ρ_0, z_0), with $\rho_0 > 0$, represent a critical circle, and write H_k for the degree-k homogeneous part of the Taylor expansion of $a - a(\rho_0, z_0)$ at (ρ_0, z_0); thus H_{k_C} is the first nonzero one. At this point (23) gives $A_{\rho\rho} + A_{zz} = 0$. If the planar Hessian of a there is nonzero, it is trace-free and nonsingular, so $k_C = 2$ and H_2 is a nonzero harmonic quadratic. If that Hessian vanishes, then $k_C \geq 3$. In translated coordinates $u = \rho - \rho_0$ and $v = z - z_0$, the term $(\rho_0 + u)^{-1} A_u$ in (23) begins in degree $k_C - 1$, whereas ΔH_{k_C} has degree $k_C - 2$. The lowest-degree part of the equation therefore gives $\Delta H_{k_C} = 0$ in this case as well. Hence, for some $c \in \mathbb{C} \setminus \{0\}$,

$$H_{k_C}(u, v) = \operatorname{Re}\big(c(u + iv)^{k_C}\big).$$

Its gradient is nonzero away from the origin and has degree $1 - k_C$. On a circle of radius r about the critical point, division by $r^{k_C - 1}$ makes the full translated gradient converge uniformly to ∇H_{k_C} as $r \to 0$. The straight-line homotopy is therefore nonvanishing for all sufficiently small r, so the local index of the full gradient is also $1 - k_C$.

The reflected meridional critical point has the same index. Indeed, with $J = \operatorname{diag}(-1, 1)$, the identity $a \circ J = a$ gives

$$\nabla_{s,z} a(Jy) = J \nabla_{s,z} a(y).$$

The local maps at (ρ_0, z_0) and $(-\rho_0, z_0)$ are thus conjugate by J in both source and target, and the two orientation reversals cancel in the Brouwer degree.

At an axial critical point $p = (0, 0, z_p)$, translation by p preserves axisymmetry and harmonicity, so by Lemma 2.2 the homogeneous Taylor terms of A at p are multiples of the zonal harmonics $Z_l(x_1, x_2, z - z_p)$. Restricting to the meridian plane turns these into the corresponding multiples of $\zeta_l(s, z - z_p)$, so the first nonconstant term of A at p is $cZ_{d_p}(x_1, x_2, z - z_p)$ with $c \neq 0$, where $d_p \geq 2$ is exactly the integer of the statement. The same small-circle homotopy, now using Lemma 7.1, gives local planar index $1 - d_p = -\mu_p$.

Equation (24) also provides a nonvanishing homotopy on a sufficiently large circle between ∇a and the gradient of its leading term. By Lemma 7.1, the degree of ∇a on that circle is therefore $1 - m$. All the zeros of ∇a lie inside it, so additivity of the planar degree over these finitely many zeros gives

$$2 \sum_C (1 - k_C) - \sum_p \mu_p = 1 - m,$$

which is equivalent to (3). Since $k_C \geq 2$ and $\mu_p \geq 1$, the number of critical circles is at most $\lfloor (m-1)/2 \rfloor$.

It remains to prove sharpness. Let $m = 2K + 1$ be odd and prescribe the axial restriction

$$A(0, 0, z) = \sum_{j=0}^{K} b_j z^{2j+1};$$

equivalently, set $A = \sum_{j=0}^{K} b_j Z_{2j+1}$. This polynomial is odd in z, so $A_\rho(\rho, 0) = 0$. The standard identity

$$(1 - t^2) P_l'(t) = l\big(P_{l-1}(t) - t P_l(t)\big)$$

and (6) give $P_{2j+1}'(0) = (2j + 1) P_{2j}(0)$. Since $\partial_z r = 0$ and $\partial_z(z/r) = \rho^{-1}$ at $(\rho, 0)$ for $\rho > 0$, it follows, with $u = \rho^2$, that

$$A_z(\rho, 0) = \sum_{j=0}^{K} (-1)^j (2j + 1) \binom{2j}{j} 4^{-j} b_j u^j.$$

The multipliers of the b_j are nonzero. We may therefore choose the b_j so that this last polynomial, denoted by $\psi(u)$, is a nonzero multiple of $\prod_{i=1}^{K} (u - u_i)$ for distinct $u_i > 0$. Its leading coefficient is nonzero, so $b_K \neq 0$ and A has degree exactly $2K + 1$. The roots produce K critical circles. At $(\rho, z) = (\sqrt{u_i}, 0)$, oddness in z gives

$$A_{\rho\rho} = A_{zz} = 0, \qquad A_{\rho z} = \partial_\rho A_z = 2\sqrt{u_i}\, \psi'(u_i) \neq 0.$$

Thus the meridional Hessian has determinant $-A_{\rho z}^2 < 0$, so every displayed circle is meridionally nondegenerate. The upper bound already proved shows that these are all the critical circles. When $K = 0$, this prescription simply means choosing $b_0 \neq 0$; then $A = b_0 Z_1 = b_0 z$ has degree one and no critical circle, attaining the bound for $m = 1$.

Now let $m = 2K$ be even. For $K = 1$, the polynomial Z_2 has no critical circle, attaining the zero upper bound. For $K \geq 2$, start with the degree-$(2K - 1)$ construction, which has $K - 1$ meridionally nondegenerate critical circles, and add τZ_{2K} with $\tau \neq 0$ sufficiently small. The implicit function theorem applied to the meridional gradient preserves at least these $K - 1$ circles, while $\tau \neq 0$ makes the resulting polynomial have degree exactly $2K$. The upper bound permits no additional circles, so equality holds. This proves sharpness for every $m \geq 1$. □

Remark 7.2. For the degree-N local model in Theorem 1.1, where N is even, Theorem 1.3 permits at most $(N - 2)/2$ critical circles before the N-fold sectoral term is introduced. Near a meridionally nondegenerate critical circle, a sufficiently small nonzero perturbation

$$P_\delta = A + \delta \rho^N \cos(N\phi)$$

produces exactly $2N$ nearby nondegenerate critical points. Indeed, the angular critical equation is $\sin(N\phi) = 0$. For each sign $\sigma = \cos(N\phi) \in \{\pm 1\}$, the meridional equations are

$$A_\rho + \delta \sigma N \rho^{N-1} = 0, \qquad A_z = 0.$$

The nonsingular meridional Hessian and the implicit function theorem give one nearby solution for each σ when $\delta \neq 0$ is sufficiently small. Each sign occurs at N angles. At every resulting critical point, the orthonormal cylindrical Hessian satisfies

$$\operatorname{Hess} P_\delta(e_\phi, e_\phi) = \rho^{-2} (P_\delta) * \phi\phi + \rho^{-1} (P * \delta) * \rho = -\delta \sigma N^2 \rho^{N-2} \neq 0,$$

while

$$\operatorname{Hess} P * \delta(e_\rho, e_\phi) = \rho^{-1}(P_\delta) * \rho\phi - \rho^{-2}(P * \delta) * \phi = 0, \qquad \operatorname{Hess} P * \delta(e_z, e_\phi) = \rho^{-1}(P_\delta)_{z\phi} = 0.$$

Here criticality gives $(P_\delta)_\rho = (P_\delta)_\phi = 0$, and the remaining mixed derivatives vanish because $\sin(N\phi) = 0$. The meridional Hessian remains nonsingular by continuity, so the full Hessian is nonsingular. Consequently the meridionally nondegenerate critical circles of a degree-N axisymmetric precursor produce at most $N(N-2)$ nearby points under this single sectoral perturbation, exactly the off-axis scale achieved above. This observation applies only to that architecture; it imposes no upper bound on other constructions.

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