

A Superlinear Lower Bound for Maxwell's Equilibrium Problem

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Abstract

Let $M(n)$ be the supremum of the number of critical points of a Morse Coulomb potential $V(x) = \sum_{i=1}^n q_i / |x - a_i|$ of n distinct *positive* point charges in $\mathbb{R}^3$. We prove that

$$M(n) \geq (3.596 \times 10^{-4} - o(1)) n^{3/2}.$$

Our main two ingredients are a twelve-wave harmonic exponential packet whose critical points contain an explicitly solvable three-dimensional lattice, and an exact positive Gauss–Legendre product cubature that realizes an arbitrary harmonic polynomial of degree D as a rescaled local limit of $(D+1)(2D+1)$ positive point sources. The packet gives $\Theta(D^3)$ nondegenerate critical points. A certified Arb ball-arithmetic computation gives the displayed numerical coefficient. A separate elementary argument proves the same $n^{3/2}$ growth bound with some positive constant, so the exponent does not depend on the computer-assisted calculation.

1 Introduction and statement

An equilibrium point of a fixed electrostatic field is a position at which the force on a test charge vanishes. For a Coulomb potential this is precisely a critical point of the potential: the conventional electric field is $-\nabla V$, and the sign does not affect its zero set. The question considered here is how many isolated equilibria can be produced by n distinct positive point charges in three-dimensional space. The answer is unknown even in its order of growth.

1.1 Notation

Fix $n \geq 2$, distinct $a_1, \dots, a_n \in \mathbb{R}^3$ and $q_1, \dots, q_n > 0$, and let

$$V(x) = \sum_{i=1}^n \frac{q_i}{|x - a_i|}, \quad x \in \mathbb{R}^3 \setminus \{a_1, \dots, a_n\}. \quad (1)$$

The potential V is harmonic off the sources. Let $\text{Crit}(V) = \{x \notin \{a_1, \dots, a_n\} : \nabla V(x) = 0\}$. We call V *Morse* if every point of $\text{Crit}(V)$ is nondegenerate. At such a point the Hessian is nonsingular and has trace $\Delta V = 0$; its eigenvalues therefore have both signs. Thus its Morse index, the number of negative Hessian eigenvalues, is 1 or 2. When $\text{Crit}(V)$ is finite, write m_1, m_2 for the respective numbers of critical points of these two indices. Unless a subscript indicates otherwise, $|\cdot|$ denotes the Euclidean norm on vectors and $\|\cdot\|$ the induced operator norm on linear or multilinear maps (in particular, matrices). We use the Euclidean inner product to identify vectors and covectors.

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Definition 1.1. $M(n)$ is the supremum of $\#\text{Crit}(V)$ over configurations as above for which $\text{Crit}(V)$ is finite and V is Morse. For an integer $D \geq 1$, let $R(D)$ be the supremum of the number of real nondegenerate critical points of a harmonic polynomial $P : \mathbb{R}^3 \rightarrow \mathbb{R}$ of degree at most D .

Thus $M(n)$ is the largest possible number of equilibria, not the number present in every configuration. The finiteness and Morse hypotheses exclude nonisolated critical sets and degenerate critical points; all the lower-bound constructions below are ultimately perturbed, when necessary, so that they lie in precisely this class.

1.2 History of known bounds

The problem goes back to Maxwell’s discussion of equipotential surfaces in his 1873 *Treatise* [13]. In the standard modern reconstruction, the quadratic estimate conventionally attributed to Maxwell is the conjectural inequality

$$M(n) \leq (n-1)^2. \quad (2)$$

The topology forces $n-1$ equilibria, and Maxwell asserted that there could be at most $(n-1)(n-2)$ additional ones. Maxwell did not supply a complete proof; Gabrielov, Novikov, and Shapiro [5] reconstruct the argument, explain its unproved geometric step, and formulate (2) as the Maxwell conjecture. Early critical-point-theoretic treatments include Kiang’s study of nondegenerate Newtonian potentials [10] and the work of Morse and Cairns, who apparently independently posed the problem of finding an upper bound [14, p. 293].

The universal topological baseline. For a positive Morse configuration, harmonicity permits only Morse indices 1 and 2, and the global degree relation is

$$m_2 - m_1 = n - 1, \quad \#\text{Crit}(V) = n - 1 + 2m_1. \quad (3)$$

Proposition 1.2 gives a self-contained proof. Consequently every admissible configuration has at least $n-1$ equilibria and

$$\#\text{Crit}(V) \equiv n - 1 \pmod{2}. \quad (4)$$

The lower bound is sharp as a statement about *every* configuration: collinear positive charges have exactly one equilibrium in each interval between consecutive sources, all of index 2. In particular $M(n) \geq n-1$, and the elementary two-charge case has $M(2) = 1$.

General upper bounds. Gabrielov, Novikov, and Shapiro [5] proved the first universal finite bound on the number of nondegenerate equilibria, without geometric restrictions, using fewnomial theory. In the present notation, their estimate is

$$M(n) \leq 4^{n^2} (3n)^{2n}.$$

Zolotov subsequently used the Thom–Milnor theorem for inverse-power potentials [19]; specializing his result to the Coulomb potential in $\mathbb{R}^3$ gives

$$M(n) \leq 5 \cdot 9^{n+3}.$$

The best currently known general estimate, due to Edelsbrunner, Fillmore, and Oliveira [4], is

$$M(n) \leq 2^n (3n-2)^3. \quad (5)$$

Their theorem is stated after fixing the strengths and all but one source and placing the remaining source generically. We record why their algebraization gives (5) directly for the Morse class in Definition 1.1. Write $a_i = (a_{i1}, a_{i2}, a_{i3})$, put $r_i = |x - a_i|$, and introduce variables u_i together with the polynomials

$$Q_i(x, u) = u_i^2 - \sum_{s=1}^3 (x_s - a_{is})^2, \quad R_j(x, u) = \sum_{i=1}^n q_i (x_j - a_{ij}) \prod_{\ell \neq i} u_\ell^3.$$

Here $1 \leq i \leq n$ and $1 \leq j \leq 3$; the two types of polynomial have degrees 2 and $3n - 2$, respectively. At a real equilibrium, take $u_i = r_i > 0$. The u -Jacobian of $Q = (Q_1, \dots, Q_n)$ is the invertible matrix $2 \operatorname{diag}(r_1, \dots, r_n)$. Eliminating u locally by $Q = 0$, the Schur complement of the Jacobian of (R, Q) is the derivative of $R(x, r(x))$. Since

$$R(x, r(x)) = - \left(\prod_{i=1}^n r_i^3 \right) \nabla V(x),$$

this Schur complement at a critical point is $-(\prod_i r_i^3) D^2 V(x)$ and is invertible. Every Morse equilibrium therefore lifts to a nonsingular isolated zero of the complexified polynomial system. Affine Bézout bounds the number of such zeros by the product of the degrees, $2^n (3n - 2)^3$, proving (5) without a genericity assumption. In particular, $M(n)$ is a finite integer for every $n \geq 2$.

Sharper results are known under additional restrictions and for small n . Every equilibrium of positive coplanar sources lies in the source plane: away from that plane, the normal component of ∇V has a fixed nonzero sign. For the resulting planar problem, Killian proved the upper bound $2^{2n-2} (3n - 2)^2$, which is not an upper bound for unrestricted spatial configurations [11]. For three charges, Gabrielov, Novikov, and Shapiro first obtained the special upper bound 12 [5]. Restricted three-charge analyses include Tsai's proof of the four-point bound for equilateral and isosceles-right configurations [16], Peretz's argument-principle derivation of a planar three-charge bound of 12 [15], and Tsai's proof that three equal positive charges have at most four equilibria [17]. Other exact or symbolic studies of special geometries include Bilogliadov's regular-polygon analysis [3] and the planar three- and four-charge bifurcation computations of Uteshev and Yashina [18]. Most recently, Gabrielov, Dm. Novikov, T. Novikov, and Shapiro proved that three arbitrary positive charges have at most six nondegenerate equilibria [6]. Three equal charges at the vertices of an equilateral triangle have four nondegenerate equilibria, so

$$4 \leq M(3) \leq 6. \tag{6}$$

Together with (4), this leaves the two possibilities $M(3) = 4$ and $M(3) = 6$. Lee and Tsai constructed four coplanar positive charges with nine nondegenerate equilibria, giving $M(4) \geq 9$ [12]. Their nine-solution parameter region is open and avoids the bifurcation locus, so the planar critical points can be chosen nondegenerate. On the source plane the mixed normal derivatives vanish and

$$\partial_{zz} V(x, y, 0) = - \sum_i \frac{q_i}{|(x, y, 0) - a_i|^3} < 0.$$

Consequently planar nondegeneracy is also nondegeneracy in $\mathbb{R}^3$.

Lower bounds. Beyond the universal baseline $M(n) \geq n - 1$, the first large families still had only linear growth. Edelsbrunner, Fillmore, and Oliveira iterated a square-antiprism construction.

For every $\ell \geq 1$, they construct $n_\ell = 8^\ell$ equal positive charges with $\frac{25}{7}(8^\ell - 1)$ distinct equilibria. The equilibria retained by the construction are nondegenerate and persist through the iteration; an arbitrarily small generic perturbation makes the complete potential Morse without losing them, placing the construction in the class of Definition 1.1 [4]. Hence

$$M(8^\ell) \geq \frac{25}{7}(8^\ell - 1), \quad \limsup_{n \rightarrow \infty} \frac{M(n)}{n} \geq \frac{25}{7}. \quad (7)$$

Notice that the first inequality is on the sparse subsequence 8^ℓ ; it does not imply a coefficient $25/7$ for every n .

Arathoon, Ball, and Kvalheim then found five positive charges with at least 24 nondegenerate equilibria. A sufficiently small perturbation of the charge strengths makes the entire critical set finite and Morse while retaining those points, and hence $M(5) \geq 24 > 16 = (5-1)^2$. This disproves the literal Maxwell inequality (2). Their iterated local bifurcation gives, for every $m \geq 0$,

$$M(3 + 2m) \geq 4 + 20m,$$

or equivalently

$$M(n) \geq \begin{cases} 10n - 26, & n \geq 3 \text{ odd}, \\ 10n - 36, & n \geq 4 \text{ even}, \end{cases} \quad (8)$$

where the even case follows from Proposition 1.3 below [2]. All previously known asymptotic constructions for the physical Coulomb exponent were linear, and our Theorem 1.4 is the first superlinear lower bound for positive Coulomb charges.

Complementary work concerns computation rather than extremal counting. Ghosh, Goldberg, and Hollender give algorithms and complexity results for finding approximate equilibria of a specified electrostatic potential [7]. This does not furnish a bound on $M(n)$, but it places the counting problem in a broader algorithmic context. For a broader perspective on equilibrium questions in classical electrostatics, including the Dyson–Lenard theorem, see the survey of Abanov, Hayford, Khavinson, and Teodorescu [1].

1.3 Two structural facts

The following propositions are well known. We present the proofs here for the sake of completeness.

Proposition 1.2 (Global degree identity). *If V is Morse with finitely many critical points then $m_2 - m_1 = n - 1$, so $\# \text{Crit}(V) = 2m_1 + (n - 1)$.*

Proof. Put $Q = \sum_i q_i$. Choose R large and $r_1, \dots, r_n > 0$ small enough that the closed balls $\bar{B}(a_i, r_i)$ are pairwise disjoint and contained in $B(0, R)$, every critical point belongs to

$$\Omega = B(0, R) \setminus \bigcup_i \bar{B}(a_i, r_i),$$

and ∇V is nonzero on $\partial\Omega$. On the outer sphere, uniformly as $R \rightarrow \infty$,

$$\nabla V(x) = -Q \frac{x}{|x|^3} + O(|x|^{-3}),$$

so the straight-line homotopy to $-Qx/|x|^3$ avoids zero for sufficiently large R . Hence the normalized field on the outer boundary is homotopic to the antipodal map of $\mathbb{S}^2$ and has degree -1 . On the sphere $x = a_i + r_i u$, parametrized by $u \in \mathbb{S}^2$ with its standard orientation,

$$\nabla V(a_i + r_i u) = -\frac{q_i}{r_i^2} u + O(1) \quad (r_i \downarrow 0),$$

uniformly in u . Its normalized field is therefore also homotopic to the antipodal map and has degree -1 in the standard sphere orientation. The orientation induced on this sphere as an inner component of $\partial\Omega$ is the opposite one, so its boundary contribution is $+1$.

The degree theorem for vector fields now identifies the degree of $\nabla V/|\nabla V|$ on the oriented boundary $\partial\Omega$ with the sum of the local indices of the zeros in Ω . A Morse critical point of index k has local index $\text{sgn det } D^2V = (-1)^k$. Consequently $-m_1 + m_2 = -1 + n$, which is the asserted identity. $\square$

Proposition 1.3 (Monotonicity). *$M(n+1) \geq M(n)$ for every $n \geq 2$.*

Proof. Start with any Morse n -charge potential V_0 having finitely many critical points, and choose a point b distinct from the sources and the critical points. Place at b a positive charge of strength t , and denote the resulting potential by

$$V_t(x) = V_0(x) + \frac{t}{|x - b|}.$$

On small, pairwise disjoint closed neighbourhoods of the critical points of V_0 that avoid b and the sources, one has $V_t \rightarrow V_0$ in C^2 as $t \downarrow 0$. The implicit function theorem therefore shows that all of those critical points persist, and remain nondegenerate, for small $t > 0$.

It remains to arrange that the entire new potential is Morse. Put $N = n + 1$ and fix the resulting positive strengths $\mathbf{q} = (q_1, \dots, q_n, t)$. Let $\mathcal{A}_N \subset (\mathbb{R}^3)^N$ be the open set of ordered, pairwise distinct source locations, and consider the open incidence manifold

$$\mathcal{I}_N = \{(\mathbf{a}, x) \in \mathcal{A}_N \times \mathbb{R}^3 : x \neq a_i \text{ for every } i\}$$

and the universal field $\mathcal{E} : \mathcal{I}_N \rightarrow \mathbb{R}^3$,

$$\mathcal{E}(\mathbf{a}, x) = - \sum_i q_i \frac{x - a_i}{|x - a_i|^3}.$$

Its derivative with respect to any one location a_i is

$$\frac{q_i}{|x - a_i|^3} (I - 3u_i u_i^\top), \quad u_i = \frac{x - a_i}{|x - a_i|},$$

whose eigenvalues are $q_i |x - a_i|^{-3} (1, 1, -2)$; hence $\mathcal{E}$ is a submersion on $\mathcal{I}_N$. Consequently $\mathcal{Z} = \mathcal{E}^{-1}(0)$ is an embedded C^∞ submanifold of $\mathcal{I}_N$ of codimension 3. Let $\pi : \mathcal{Z} \rightarrow \mathcal{A}_N$ be the restriction of the projection $(\mathbf{a}, x) \mapsto \mathbf{a}$. By Sard's theorem applied to π , the set of critical values of π has measure zero in $\mathcal{A}_N$; and $\mathbf{a}$ is a regular value of π precisely when the slice $x \mapsto \mathcal{E}(\mathbf{a}, x)$ is transverse to 0, that is, when every equilibrium is nondegenerate. Indeed, at $(\mathbf{a}, x) \in \mathcal{Z}$ one has

$$T_{(\mathbf{a}, x)}\mathcal{Z} = \{(\dot{\mathbf{a}}, \dot{x}) : D_{\mathbf{a}}\mathcal{E} \dot{\mathbf{a}} + D_x\mathcal{E} \dot{x} = 0\}.$$

Because $D_{\mathbf{a}}\mathcal{E}$ is surjective, the restriction of the projection to this tangent space is surjective if and only if $D_x\mathcal{E}$ is surjective. (We use Sard directly rather than a parametric-transversality theorem for a product family, because $\mathcal{I}_N$ is an open subset of $\mathcal{A}_N \times \mathbb{R}^3$ and not a product; π is smooth on the manifold $\mathcal{Z}$, which is all Sard requires. Second countability of $\mathcal{Z}$ makes the measure-zero conclusion global, so no compact exhaustion is needed.) The complement of a measure-zero set is dense, so we may choose such a configuration arbitrarily close to the current one. At a zero of the

slice, its derivative in x is D^2V ; transversality therefore says precisely that the new potential is Morse. The already-persisting critical points survive this further small perturbation.

For completeness, the critical set of this Morse potential is finite. Let C be the convex hull of the sources. If $x \notin C$, strict separation gives a vector v with $v \cdot (x - a_i) > 0$ for every i , and hence

$$v \cdot \mathcal{E}(\mathbf{a}, x) = - \sum_i q_i \frac{v \cdot (x - a_i)}{|x - a_i|^3} < 0.$$

Thus every equilibrium lies in C . Moreover, uniformly for $u \in \mathbb{S}^2$,

$$\mathcal{E}(\mathbf{a}, a_i + ru) = -\frac{q_i}{r^2}u + O(1) \quad (r \downarrow 0),$$

so there are radii $r_i > 0$ for which the punctured balls $B(a_i, r_i) \setminus \{a_i\}$ contain no equilibria. Consequently

$$\text{Crit}(V) = \mathcal{E}(\mathbf{a}, \cdot)^{-1}(0) \cap K, \quad K = C \setminus \bigcup_i B(a_i, r_i).$$

The set K is compact, and the displayed zero set is closed in K , hence compact. Because transversality makes it a zero-dimensional submanifold, it is also discrete and therefore finite.

This construction starts from every admissible n -charge potential and preserves all of its critical points, so taking suprema gives $M(n+1) \geq M(n)$. The same transversality and compactness argument, applied without first adding a charge, also shows that the class defining $M(n)$ is nonempty for every $n \geq 2$. $\square$

1.4 The main theorem and proof strategy

Theorem 1.4 (Main).

$$\liminf_{n \rightarrow \infty} \frac{M(n)}{n^{3/2}} \geq c_0, \quad c_0 = \frac{1.0172 \times 10^{-3}}{2\sqrt{2}} = 3.596345089 \dots \times 10^{-4}.$$

Equivalently,

$$\boxed{M(n) \geq (c_0 - o(1))n^{3/2}.$$

Here the $o(1)$ term may be taken nonnegative and tends to zero as $n \rightarrow \infty$. In particular $M(n) = \Omega(n^{3/2})$, and the positivity of the implicit constant in this qualitative conclusion has a proof that uses no numerical computation.

Combining Theorem 1.4 with (5) gives the current rigorous asymptotic window

$$(c_0 - o(1))n^{3/2} \leq M(n) \leq 2^n(3n-2)^3. \quad (9)$$

Although (2) is false, its quadratic scale remains a useful benchmark. The new $n^{3/2}$ lower bound does not reach that scale, and the gap in (9) remains substantial.

The proof occupies Sections 2–6 and has two main stages. First, a twelve-wave harmonic exponential packet is chosen so that its critical set contains an explicitly solvable three-dimensional lattice. Quantitative persistence and Taylor truncation turn the packet into degree- D harmonic polynomials and prove

$$R(D) \geq (1.0172 \times 10^{-3} - o(1))D^3.$$

Second, an exact positive Gauss–Legendre product cubature on $\mathbb{S}^2$ realizes the local jet of every such polynomial using

$$N_D = (D + 1)(2D + 1)$$

positive Coulomb sources. Positivity of the cubature weights is what enforces positivity of every physical charge. Rescaling and persistence give $M(N_D) \geq R(D)$. When $n = N_D$, the exact identity $D = (\sqrt{8n+1} - 3)/4$ gives $D^3 = (1 + o(1))n^{3/2}/(2\sqrt{2})$ and hence the coefficient c_0 above. For every $n \geq 6$, put

$$D_n = \left\lfloor \frac{\sqrt{8n+1} - 3}{4} \right\rfloor$$

so that $N_{D_n} \leq n$. Proposition 1.3 then passes the bound from N_{D_n} to n , while $D_n = (1+o(1))\sqrt{n/2}$.

The displayed numerical coefficient comes from a certified lower bound for an explicit three-dimensional volume. The certificate uses Arb midpoint-radius ball arithmetic [9]; all comparisons needed by the proof are interval-certified rather than inferred from floating-point output. A separate elementary positive-volume argument proves $R(D) \geq cD^3$ for some $c > 0$, so the exponent $3/2$ is independent of the computer-assisted estimate. The certificate files are publicly available at github.com/shivakintali/Maxwell-Superlinear

2 Persistence and realization

2.1 Abstract persistence

For a function f of class C^m on a neighbourhood of a compact set K , set

$$\|f\| * C^m(K) = \max_{0 \leq j \leq m} \sup_{X \in K} \|D^j f(X)\|,$$

where $\|D^0 f(X)\| = |f(X)|$ and the higher derivatives carry their induced Euclidean operator norms.

Theorem 2.1 (Persistence). *Let $K \subset \mathbb{R}^3$ be nonempty and compact, and let $H \in C^2$ on a neighbourhood of K with*

(P1) *every critical point of H in K nondegenerate,*

(P2) *no critical point of H on ∂K .*

Put $Z = \text{Crit}(H) \cap K$, $R = \#Z < \infty$. Then there is $\eta > 0$, depending only on H and K , such that every $F \in C^2$ on a neighbourhood of K with $\|F - H\|_{C^2(K)} < \eta$ has exactly R critical points in K , all nondegenerate, one near each point of Z , with the same corresponding Morse indices. Moreover the correspondence may be localized quantitatively: given any $\varepsilon > 0$, the number η may be chosen (depending on H , K and ε) so that the critical point of F associated with $p \in Z$ lies in $B(p, \varepsilon)$ for every p .

Proof. Z is closed in K and hence compact. Its points are isolated because their Hessians are nonsingular. If Z were infinite, compactness would give an accumulation point in Z , contradicting this isolation; hence Z is finite. If $Z = \emptyset$, let $m = \min_K |\nabla H| > 0$. Whenever $\|F - H\|_{C^2(K)} < m$, the inequality $|\nabla F| \geq m - \|F - H\|_{C^2(K)} > 0$ on K gives the conclusion.

Assume henceforth that $Z \neq \emptyset$. Condition (P2) places every point of Z in $\text{int } K$. For $p \in Z$ put $\sigma_p = \sigma_{\min}(D^2 H(p)) > 0$ (the smallest singular value) and choose $r_p > 0$ so that (i) the

balls $\bar{B}(p, r_p)$ are pairwise disjoint, are contained in $\text{int } K$, and contain no other point of Z , (ii) $\|D^2H(X) - D^2H(p)\| < \sigma_p/3$ on $\bar{B}(p, r_p)$. Let $K' = K \setminus \bigcup_{p \in Z} B(p, r_p)$, compact and critical-point free, so $|\nabla H| \geq \delta' > 0$ there. Put

$$\delta = \min_p \min_{|X-p|=r_p} |\nabla H(X)| > 0, \quad \eta = \min\left(\frac{1}{2}\delta, \frac{1}{2}\delta', \min_p \frac{\sigma_p}{3}\right).$$

Let $\|F - H\|_{C^2(K)} < \eta$.

Existence and count. On $\partial B(p, r_p)$ the homotopy $G_t = (1-t)\nabla H + t\nabla F$ satisfies $|G_t| \geq |\nabla H| - |\nabla F - \nabla H| > \delta - \eta \geq \delta/2$. The point p is the only zero of ∇H in $B(p, r_p)$, so invariance and the local formula for Brouwer degree give

$$\deg(\nabla F, B(p, r_p), 0) = \deg(\nabla H, B(p, r_p), 0) = \text{sgn det } D^2H(p) = \pm 1.$$

Thus ∇F has a zero in $B(p, r_p)$.

Uniqueness in each ball. For $X \in \bar{B}(p, r_p)$, $\|D^2F(X) - D^2H(p)\| < \sigma_p/3 + \sigma_p/3$, so for $u, v \in \bar{B}(p, r_p)$

$$\begin{aligned} |\nabla F(u) - \nabla F(v)| &\geq |D^2H(p)(u-v)| \\ &\quad - \int_0^1 \|D^2F(v+t(u-v)) - D^2H(p)\| |u-v| dt \\ &\geq (\sigma_p - \frac{2}{3}\sigma_p) |u-v|, \end{aligned}$$

so ∇F is injective on the ball and the zero q_p is unique.

Nondegeneracy and index. $\|D^2F(q_p) - D^2H(p)\| < \frac{2}{3}\sigma_p < \sigma_p$, so $D^2F(q_p)$ is invertible. The same perturbation estimate shows that every matrix on the segment

$$t \mapsto D^2H(p) + t(D^2F(q_p) - D^2H(p)), \quad 0 \leq t \leq 1,$$

is invertible. These matrices are symmetric, so no eigenvalue can change sign along the segment; q_p and p therefore have the same Morse index.

No further critical points. On K' , $|\nabla F| \geq \delta' - \eta \geq \frac{1}{2}\delta' > 0$. Hence the points q_p , $p \in Z$, are all the critical points of F in K .

Localization. Given $\varepsilon > 0$, the radii r_p above may in addition be required to satisfy $r_p \leq \varepsilon$, since conditions (i) and (ii) persist under shrinking. The resulting η then forces $q_p \in B(p, r_p) \subseteq B(p, \varepsilon)$, which is the localized statement. $\square$

Remark 2.2. In every application below F and H are harmonic. If $K \subset \text{int } K_1$ and both functions are harmonic on a neighbourhood of the compact set K_1 , the interior estimates give, for every $m \geq 0$,

$$\|F - H\| * C^m(K) \leq C * m, K, K_1 \|F - H\|_{C^0(K_1)}.$$

Thus locally uniform convergence of harmonic functions gives C^∞ convergence on compact subsets, which is the form used below.

2.2 Quantitative persistence

Lemma 2.3. *Let Φ be of class C^3 on a neighbourhood of $\bar{B}(p, r)$, with $\nabla\Phi(p) = 0$, $H = D^2\Phi(p)$, and $\sigma = \sigma_{\min}(H) > 0$. Suppose that, for some $K > 0$,*

$$\|D^3\Phi(X)[v, v, \cdot]\| \leq K |v|^2 \quad (X \in \bar{B}(p, r), v \in \mathbb{R}^3).$$

If $0 < r \leq \sigma/K$, then

$$|\nabla\Phi(X)| \geq \sigma r - \frac{K}{2}r^2 \geq \frac{\sigma r}{2} =: \delta \quad \text{whenever } |X - p| = r,$$

and every continuous map $G : \bar{B}(p, r) \rightarrow \mathbb{R}^3$ with $\sup_{\bar{B}(p, r)} |G - \nabla\Phi| < \delta$ has $\deg(G, B(p, r), 0) = \operatorname{sgn} \det H = \pm 1$; in particular G vanishes somewhere in $B(p, r)$.

Proof. Taylor's formula with integral remainder gives

$$\nabla\Phi(p + v) = Hv + \int_0^1 (1-s) D^3\Phi(p + sv)[v, v, \cdot] ds.$$

Consequently $|\nabla\Phi(p + v)| \geq \sigma|v| - \frac{K}{2}|v|^2$, which at $|v| = r$ is at least $\sigma r - Kr^2/2 \geq \sigma r/2$. On $\partial B(p, r)$,

$$|(1-t)\nabla\Phi(X) + tH(X-p)| \geq \sigma r - \frac{K}{2}r^2 > 0,$$

so $\nabla\Phi$ is homotopic there to the invertible linear field $H(X-p)$. It follows that

$$\deg(\nabla\Phi, B(p, r), 0) = \operatorname{sgn} \det H.$$

For the final homotopy, on $\partial B(p, r)$ one has

$$|(1-t)\nabla\Phi + tG| \geq \delta - \sup_{\bar{B}(p, r)} |G - \nabla\Phi| > 0.$$

Thus the degrees of G and $\nabla\Phi$ agree. $\square$

Corollary 2.4. *If $p_1, \dots, p_R$ are nondegenerate critical points of Φ with disjoint balls as in Lemma 2.3, and if G is continuous on their union with $\sup_{\bigcup_i \bar{B}(p_i, r_i)} |G - \nabla\Phi| < \min_i \delta_i$, then G has at least R zeros. If $G = \nabla P$ for a C^2 function P defined on a neighbourhood of the union of the balls, and all its critical points in that union are nondegenerate, each ball contains at least one nondegenerate critical point of P .*

Proof. Lemma 2.3 gives nonzero degree, and hence at least one zero of G , in each ball. The balls are disjoint, so these zeros are distinct. Under the final hypothesis, every critical point of P in these balls is nondegenerate. $\square$

2.3 From a harmonic jet to positive charges

Let L_l denote the Legendre polynomial normalized by $L_l(1) = 1$. For $a \neq 0$ and $|x| < |a|$, the Laplace expansion is

$$\frac{1}{|x-a|} = \frac{1}{|a|} \sum_{l \geq 0} Z_l(b; x), \quad b = \frac{a}{|a|^2}, \quad Z_l(b; x) = |b|^l |x|^l L_l(\hat{b} \cdot \hat{x}).$$

The series converges absolutely and locally uniformly in the ball $|x| < |a|$. Here hats denote unit vectors, and Z_l is understood by its polynomial continuation whenever $b = 0$ or $x = 0$. The solid zonal harmonic Z_l is a polynomial, homogeneous of degree l in each of b and x . Write $\mathcal{H}_l$ for the $(2l+1)$ -dimensional space of harmonic homogeneous polynomials of degree l on $\mathbb{R}^3$.

Let $d\sigma$ be normalized surface measure on $\mathbb{S}^2$. The spherical-harmonic addition theorem gives the reproducing identity

$$(2l+1) \int_{\mathbb{S}^2} Z_l(b; \omega) h(\omega) d\sigma(\omega) = h(b), \quad b \in \mathbb{R}^3, \quad h \in \mathcal{H}_l. \quad (10)$$

With $w_i = q_i/|a_i|$, a finite Coulomb potential has, for $|x| < \min_i |a_i|$, the expansion

$$V(x) = \sum_{l \geq 0} V_l(x), \quad V_l(x) = \sum_i w_i Z_l(b_i; x), \quad (11)$$

where $b_i = a_i/|a_i|^2$. Taking the inner product of $V_l|_{\mathbb{S}^2}$ with an arbitrary $h \in \mathcal{H}_l$ in (10) gives

$$(2l+1) \int_{\mathbb{S}^2} V_l(\omega) h(\omega) d\sigma(\omega) = \sum_i w_i h(b_i).$$

Since $V_l \in \mathcal{H}_l$, it follows that

$$V_l \equiv 0 \iff \sum_i w_i h(b_i) = 0 \quad \text{for every } h \in \mathcal{H}_l. \quad (12)$$

Lemma 2.5 (Coulomb realization). *Let $D \geq 1$, let $b_1, \dots, b_n \in \mathbb{R}^3 \setminus \{0\}$ be distinct, put $\beta = \max_i |b_i|$, and define*

$$W := \bigoplus_{l=1}^D \mathcal{H}_l, \quad \dim W = \sum_{l=1}^D (2l+1) = D^2 + 2D.$$

Define the linear map

$$A : \mathbb{R}^n \rightarrow W, \quad A(w) = \left(x \mapsto \sum_i w_i Z_l(b_i; x) \right) * l = 1^D.$$

Assume:

(R1) A is surjective;

(R2) there is $w^* \in \mathbb{R}_{>0}^n$ with $A(w^*) = (0, \dots, 0, h_D)$ and $h_D \neq 0$.

Given arbitrary $h_l \in \mathcal{H}_l$ ($1 \leq l \leq D-1$), use (R1) to choose $u_l \in \mathbb{R}^n$ with $A(u_l)$ equal to h_l in degree l and 0 elsewhere, and set

$$w(\varepsilon) = w^* + \sum *l = 1^{D-1} \varepsilon^{D-l} u_l, \quad a_i = \frac{b_i}{|b_i|^2}, \quad q_i(\varepsilon) = \frac{w_i(\varepsilon)}{|b_i|},$$

and let

$$V_\varepsilon(x) = \sum_{i=1}^n \frac{q_i(\varepsilon)}{|x - a_i|}.$$

Then $q_i(\varepsilon) > 0$ for all small $\varepsilon > 0$ and, for every $R_0 > 0$,

$$\varepsilon^{-D} (V_\varepsilon(\varepsilon X) - V_\varepsilon(0)) \longrightarrow H(X) := \sum_{l=1}^D h_l(X) \quad \text{in } C^\infty \text{ on } |X| \leq R_0.$$

Proof. The inversion $b \mapsto b/|b|^2$ is an involution of $\mathbb{R}^3 \setminus \{0\}$, so the source locations a_i are distinct. Moreover, $w_i(\varepsilon) \rightarrow w_i^* > 0$ for every i ; since there are finitely many coordinates, all $w_i(\varepsilon)$, and hence all $q_i(\varepsilon)$, are positive for sufficiently small $\varepsilon > 0$. The identities $|a_i| = |b_i|^{-1}$ and $q_i(\varepsilon)/|a_i| = w_i(\varepsilon)$ show that the $w_i(\varepsilon)$ are exactly the weights in (11).

By linearity of A , the degree- l term $V_l^{(\varepsilon)}$ in that expansion is

$$V_l^{(\varepsilon)} = \varepsilon^{D-l} h_l, \quad 1 \leq l \leq D,$$

where the case $l = D$ is (R2). Since $V_0^{(\varepsilon)} = V_\varepsilon(0)$ and every $V_l^{(\varepsilon)}$ is homogeneous of degree l , it follows that

$$V_\varepsilon(\varepsilon X) - V_\varepsilon(0) = \varepsilon^D H(X) + \sum_{l>D} \varepsilon^l V_l^{(\varepsilon)}(X).$$

For fixed R_0 , this expansion is valid on $|X| \leq 2R_0$ once $2\varepsilon\beta R_0 < 1$. Since $|L_l(t)| \leq 1$ for $-1 \leq t \leq 1$ and $w(\varepsilon) \rightarrow w^*$,

$$|Z_l(b; x)| \leq (|b| |x|)^l, \quad \sum_i |w_i(\varepsilon)| \leq C$$

for all sufficiently small ε , with C independent of l and ε . In particular, $|V_l^{(\varepsilon)}(X)| \leq C(\beta R_0)^l$ on $|X| \leq R_0$. Thus, after division by ε^D , the tail on $|X| \leq R_0$ is bounded by

$$C(\beta R_0)^D \sum_{k \geq 1} (\varepsilon \beta R_0)^k,$$

which tends to 0 as $\varepsilon \downarrow 0$.

Set $T_\varepsilon(X) = \varepsilon^{-D}(V_\varepsilon(\varepsilon X) - V_\varepsilon(0)) - H(X)$. The preceding estimate with $2R_0$ in place of R_0 shows that $T_\varepsilon \rightarrow 0$ uniformly on $\bar{B}(0, 2R_0)$. The singularities a_i/ε in the rescaled coordinates lie outside this ball when ε is small. Both the rescaled potential and H are harmonic there, so T_ε is harmonic there. The interior estimates imply, for every multi-index α ,

$$\sup_{|X| \leq R_0} |D^\alpha T_\varepsilon(X)| \leq C_{\alpha, R_0} \sup_{|X| \leq 2R_0} |T_\varepsilon(X)| \rightarrow 0.$$

This is the asserted C^∞ convergence. □

3 $R(D) = \Theta(D^3)$

3.1 The packet and its critical lattice

For $\lambda > 0$ set

$$\Phi_\lambda(x, y, z) = \cos x \cos y \cosh(\sqrt{2}z) + \lambda \cos(\sqrt{2}z) \cosh(\sqrt{2}x). \quad (13)$$

Lemma 3.1. $\Delta \Phi_\lambda = 0$.

Proof. The three second derivatives contribute factors $-1, -1, +2$ to the first term and $+2, 0, -2$ to the second; both sum to 0. □

As a null-exponential packet $\Phi_\lambda = \sum_\nu A_\nu e^{k_\nu \cdot X}$ it has twelve waves:

number	k	A_ν
8	$(\pm i, \pm i, \pm \sqrt{2})$	$1/8$
4	$(\pm \sqrt{2}, 0, \pm i \sqrt{2})$	$\lambda/4$

All signs in each row are chosen independently. Every wave vector satisfies $k \cdot k = 0$ and $|k| = 2$, where $|\cdot|$ is the standard Hermitian norm on $\mathbb{C}^3$ in the latter expression.

Proposition 3.2. Put $x_j = (j + \frac{1}{2})\pi$, $z_m = m\pi/\sqrt{2}$ ($j, m \in \mathbb{Z}$) and

$$\gamma_{jm} = (-1)^{j+m} \frac{\sqrt{2} \lambda \sinh(\sqrt{2}x_j)}{\cosh(m\pi)}. \quad (14)$$

If $|\gamma_{jm}| \leq 1$ then every point (x_j, y, z_m) with $\cos y = \gamma_{jm}$ is a critical point of Φ_λ , with Hessian

$$\begin{pmatrix} A & B & C \\ B & 0 & 0 \\ C & 0 & -A \end{pmatrix}, \quad \begin{aligned} A &= 2(-1)^m \lambda \cosh(\sqrt{2}x_j), \\ B &= (-1)^j \sin(y) \cosh(m\pi), \\ C &= (-1)^{j+1} \sqrt{2} \cos(y) \sinh(m\pi), \end{aligned} \quad (15)$$

which is traceless and satisfies

$$\det D^2 \Phi_\lambda(x_j, y, z_m) = AB^2 = 2(-1)^m \lambda \sin^2(y) \cosh^2(m\pi) \cosh(\sqrt{2}x_j). \quad (16)$$

The point is nondegenerate iff $\sin y \neq 0$, equivalently iff $|\gamma_{jm}| < 1$. When it is nondegenerate, its Morse index is 2 if m is even and 1 if m is odd.

Proof. $\cos x_j = 0$ makes

$$\partial_y \Phi_\lambda = -\cos x \sin y \cosh(\sqrt{2}z)$$

vanish and reduces

$$\begin{aligned} \partial_z \Phi_\lambda &= \sqrt{2} \cos x \cos y \sinh(\sqrt{2}z) - \sqrt{2} \lambda \sin(\sqrt{2}z) \cosh(\sqrt{2}x) \\ &= -\sqrt{2} \lambda \sin(\sqrt{2}z) \cosh(\sqrt{2}x_j). \end{aligned}$$

The latter vanishes exactly at $z = z_m$. At this value,

$$\begin{aligned} \partial_x \Phi_\lambda &= -\sin x \cos y \cosh(\sqrt{2}z) + \sqrt{2} \lambda \cos(\sqrt{2}z) \sinh(\sqrt{2}x) \\ &= -(-1)^j \cos y \cosh(m\pi) + (-1)^m \sqrt{2} \lambda \sinh(\sqrt{2}x_j), \end{aligned}$$

whose vanishing is (14). The Hessian entries follow by direct differentiation using $\cos x_j = 0$, $\sin x_j = (-1)^j$, $\cos(\sqrt{2}z_m) = (-1)^m$, $\sin(\sqrt{2}z_m) = 0$; expanding (15) along the second row gives $\det = -B(-AB) = AB^2$. A traceless symmetric 3×3 matrix cannot have all eigenvalues of one sign, so $\det > 0$ forces signature $(+, -, -)$ (index 2) and $\det < 0$ forces $(-, +, +)$ (index 1). By (16), the sign of $\det$ is that of $(-1)^m$. $\square$

For each strictly admissible pair (j, m) , if $\alpha_{jm} = \arccos \gamma_{jm} \in (0, \pi)$, then the solutions in y are $\pm \alpha_{jm} + 2\pi\mathbb{Z}$. Thus the set has spacing π in x , spacing $\pi/\sqrt{2}$ in z , and two y -roots per period 2π . The resulting local lattice-density factor, used below on bounded rescaled regions, is

$$\rho = \frac{1}{\pi} \cdot \frac{\sqrt{2}}{\pi} \cdot \frac{1}{\pi} = \frac{\sqrt{2}}{\pi^3}. \quad (17)$$

The elementary estimates for $\sinh$ and $\cosh$ give, uniformly in j, m ,

$$\log |\gamma_{jm}| = \sqrt{2}(|x_j| - |z_m|) + \log(\sqrt{2}\lambda) + O(1).$$

Thus the boundary of the admissible set is described, to within a bounded displacement, by the cone

$$|x| \leq |z| + \frac{1}{\sqrt{2}} \log \frac{1}{\sqrt{2}\lambda}. \quad (18)$$

3.2 Explicit stability constants

Restrict to the selected set $\{(j, m) : |\gamma_{jm}| \leq \frac{1}{2}\}$, so $|\sin y| \geq \frac{\sqrt{3}}{2}$. Write

$$U = \cosh(\sqrt{2}x_j) \geq \cosh(\pi/\sqrt{2}), \quad V = \cosh(m\pi) \geq 1.$$

The selection inequality gives

$$\frac{\lambda U}{V} \leq \frac{1}{2\sqrt{2}} \coth(\sqrt{2}|x_j|) \leq \frac{1}{2\sqrt{2}} \coth(\pi/\sqrt{2}) < \frac{3}{8}.$$

The last bound is elementary: $\pi > 3$ and $\sqrt{2} < 3/2$ give $\pi/\sqrt{2} > 2$; also $e > 8/3$ gives $\coth 2 < 21/20$, while $\sqrt{2} > 7/5$ gives $(21/20)/(2\sqrt{2}) < 3/8$. Using $\cos y = \gamma_{jm}$ and $|\sinh(m\pi)| \leq V$, one has $|C| = \sqrt{2}|\gamma_{jm}| |\sinh(m\pi)| \leq V/\sqrt{2}$. Consequently $\frac{\sqrt{3}}{2}V \leq |B| \leq V$, and $|A| = 2\lambda U < 3V/4$. Let $H_p = D^2\Phi_\lambda(p)$ and $\sigma_p = \sigma_{\min}(H_p)$, and let $\|\cdot\|_F$ denote the Frobenius norm. If u, v are the two eigenvalues other than one of absolute value σ_p , then $2|uv| \leq u^2 + v^2 \leq \|H_p\|_F^2$. Hence $|\det H_p| \leq \sigma_p \|H_p\|_F^2/2$. Since $\|H_p\|_F^2 = 2(A^2 + B^2 + C^2)$, it follows that

$$\sigma_p \geq \frac{|A|B^2}{A^2 + B^2 + C^2} > \frac{(2\lambda U)^3 V^2}{(33/16)V^2} = \frac{8}{11}\lambda U. \quad (19)$$

The exponential representation gives

$$\|D^3\Phi_\lambda(X)[v, v, \cdot]\| \leq \left(\sum_\nu |A_\nu| |k_\nu|^3 e^{\operatorname{Re}(k_\nu \cdot X)} \right) |v|^2.$$

The majorant in parentheses has the exact closed form

$$K_\lambda(X) = 8 \cosh(\sqrt{2}z) + 8\lambda \cosh(\sqrt{2}x), \quad (20)$$

and on $\bar{B}(p, r)$ with $r \leq 0.1$,

$$K_\lambda(X) \leq e^{\sqrt{2}r} 8(V + \lambda U) < \frac{7}{6} \cdot 8 \cdot \frac{11}{8}V = \frac{77}{6}V.$$

Here we used $\cosh(s+t) \leq e^{|t|} \cosh s$. Also, $e^{\sqrt{2}/10} < 7/6$: indeed, $\sqrt{2} < 3/2$ and $\log(1+t) > t - t^2/2$ at $t = 1/6$ give $\sqrt{2}/10 < 3/20 < 11/72 < \log(7/6)$. Set

$$r_p = \frac{48}{847} \frac{\lambda U}{V} < \frac{18}{847} < 0.0213.$$

The preceding bound for K_λ is therefore valid on $\bar{B}(p, r_p)$. Moreover, if $K_p = \sup_{X \in \bar{B}(p, r_p)} K_\lambda(X)$, then

$$\frac{\sigma_p}{K_p} > \frac{(8/11)\lambda U}{(77/6)V} = \frac{48}{847} \frac{\lambda U}{V} = r_p.$$

Thus the radius hypothesis in Lemma 2.3 holds, and that lemma gives the boundary margin

$$\delta_p := \frac{\sigma_p r_p}{2} > \frac{192}{9317} \frac{\lambda^2 U^2}{V} > \frac{1}{50} \frac{\lambda^2 U^2}{V}. \quad (21)$$

The balls are pairwise disjoint. Indeed, under $|\gamma_{jm}| \leq 1/2$ one has $\alpha_{jm} \in [\pi/3, 2\pi/3]$, so the two interlaced y -sequences have successive gaps $2\alpha_{jm}$ and $2\pi - 2\alpha_{jm}$, both at least $2\pi/3$. The minimal separation between distinct selected centres is therefore $\min(\pi, \pi/\sqrt{2}, 2\pi/3) = 2\pi/3 > 2$, whereas $2r_p < 36/847 < 0.043$.

3.3 The Taylor remainder

Let T_D truncate the Taylor series at 0 at total degree D . Comparing homogeneous degrees in $\Delta\Phi_\lambda = 0$ shows that every homogeneous Taylor term is harmonic; hence $T_D\Phi_\lambda$ is a harmonic polynomial. Put $E = \Phi_\lambda - T_D\Phi_\lambda$, $S_D(u) = \sum_{n \geq D} u^n/n!$, $W_1 = |x| + |y| + \sqrt{2}|z|$, and $W_2 = \sqrt{2}(|x| + |z|)$.

Lemma 3.3 (Taylor remainder).

$$|\nabla E(X)| \leq 2S_D(W_1(X)) + 2\lambda S_D(W_2(X)). \quad (22)$$

Proof. For one wave $Ae^{k \cdot X}$ the degree- D Taylor polynomial at the origin is $A \sum_{n \leq D} (k \cdot X)^n/n!$, whose gradient is $Ak \sum_{n \leq D-1} (k \cdot X)^n/n!$; subtracting it from $\nabla(Ae^{k \cdot X}) = Ak e^{k \cdot X}$ leaves the gradient remainder

$$Ak \sum_{n \geq D} \frac{(k \cdot X)^n}{n!} = Ak S_D(k \cdot X)$$

(note the index shift: the gradient of the degree- D truncation is the degree- $(D-1)$ truncation of the gradient, so the remainder still starts at $n = D$). Here $k \in \mathbb{C}^3$ and $X \in \mathbb{R}^3$, and $|k \cdot X| \leq |k| |X|$ by the Cauchy–Schwarz inequality for the Hermitian inner product on $\mathbb{C}^3$, $|k|$ denoting the Hermitian norm; more precisely we use the sharper coordinatewise bounds $|k \cdot X| \leq W_1(X)$ for each of the first eight waves and $|k \cdot X| \leq W_2(X)$ for each of the remaining four. Since S_D has nonnegative coefficients, $|S_D(z)| \leq S_D(|z|)$ for every $z \in \mathbb{C}$, and S_D is increasing on $[0, \infty)$; hence $|S_D(k \cdot X)| \leq S_D(W_i(X))$ for the respective groups. As $|k| = 2$ for all twelve waves, summing absolute values gives the two prefactors $8 \cdot (1/8) \cdot 2 = 2$ and $4 \cdot (\lambda/4) \cdot 2 = 2\lambda$, proving (22). $\square$

Lemma 3.4. *A selected point p satisfies $\deg(\nabla T_D\Phi_\lambda, B(p, r_p), 0) = \pm 1$ provided*

$$2S_D(W_1(p) + 0.043) + 2\lambda S_D(W_2(p) + 0.043) < \frac{1}{50} \frac{\lambda^2 \cosh^2(\sqrt{2}x_j)}{\cosh(m\pi)}. \quad (23)$$

Proof. The Euclidean Lipschitz constants of W_1 and W_2 are respectively $\sqrt{1^2 + 1^2 + (\sqrt{2})^2} = 2$ and $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$. Hence, for $X \in \bar{B}(p, r_p)$,

$$W_i(X) \leq W_i(p) + 2r_p < W_i(p) + 0.043 \quad (i = 1, 2).$$

Lemma 3.3 and (23) therefore give

$$\sup_{\bar{B}(p, r_p)} |\nabla T_D\Phi_\lambda - \nabla\Phi_\lambda| < \frac{1}{50} \frac{\lambda^2 U^2}{V} < \delta_p.$$

The final assertion of Lemma 2.3 now gives the stated degree. $\square$

3.4 The limiting region

Define $\varphi(0) = -\infty$ and, for $u > 0$,

$$\varphi(u) = \begin{cases} 1 + \log u, & 0 < u \leq 1, \\ u, & u \geq 1. \end{cases}$$

Lemma 3.5 (Exponential rate of the Taylor tail). *For $t \geq 0$ let $S_D(t) = \sum_{k \geq D} t^k/k!$. Then*

$$\frac{1}{D} \log S_D(Du) \longrightarrow \varphi(u)$$

locally uniformly for $u \in (0, \infty)$. Replacing Du by $Du + c_D$, where (c_D) is bounded and the arguments are nonnegative, gives the same locally uniform limit on $(0, \infty)$. In addition, for every $L < \infty$ there is a sequence $\varepsilon_D(L) \rightarrow 0$ such that, for $0 < u \leq L$,

$$\frac{1}{D} \log S_D(Du) \leq \varphi(u) + \varepsilon_D(L). \quad (24)$$

At $u = 0$, both $D^{-1} \log S_D(0)$ and $\varphi(0)$ are understood as $-\infty$. Moreover,

$$\frac{1}{D} \log \cosh(\sqrt{2}Du) \longrightarrow \sqrt{2}|u|$$

uniformly for u in bounded intervals.

Proof. Fix $\eta > 0$. If $0 < u \leq 1 - \eta$, consecutive terms in the tail have ratio at most u , so

$$\frac{(Du)^D}{D!} \leq S_D(Du) \leq \frac{1}{1-u} \frac{(Du)^D}{D!}.$$

Stirling's formula gives the limit $1 + \log u$, uniformly on compact subintervals of $(0, 1)$. If $u \geq 1 + \eta$, then for all sufficiently large D the index $k = \lfloor Du \rfloor$ belongs to the tail. This single term, together with the upper bound $S_D(Du) \leq e^{Du}$, gives the limit u , again by Stirling's formula, uniformly when u is also bounded above. On $[1 - \eta, 1 + \eta]$, monotonicity of $S_D(Du)$ traps its logarithmic rate between the rates at the two endpoints. Both limiting endpoint values tend to $\varphi(1) = 1$ as $\eta \downarrow 0$, and φ is continuous at 1. This proves local uniformity across $u = 1$.

If (c_D) is bounded, then $Du + c_D = D(u + c_D/D)$. On each compact subset of $(0, \infty)$ the shifted arguments remain in a slightly larger compact subset, so the local uniform convergence just proved and the uniform continuity of φ give the same limit.

For the additional uniform upper bound, if $0 < u \leq 1$, every consecutive-term ratio is at most $D/(D+1)$, and therefore

$$S_D(Du) \leq (D+1) \frac{(Du)^D}{D!}.$$

After division by D and use of Stirling's formula, the right side has rate

$$1 + \log u + O\left(\frac{\log D}{D}\right),$$

where the error is independent of $u \in (0, 1]$. For $1 \leq u \leq L$, the bound $S_D(Du) \leq e^{Du}$ gives (24) with zero error. This proves the asserted uniform upper estimate; the assertion at $u = 0$ is exact. Finally,

$$\frac{1}{2}e^{\sqrt{2}D|u|} \leq \cosh(\sqrt{2}Du) \leq e^{\sqrt{2}D|u|},$$

which proves the last assertion. $\square$

Take $\lambda = \lambda_D = e^{-\sqrt{2}\sigma D}$ with $\sigma \geq 0$ fixed, and rescale $Y = X/D$. Set

$$w_1(Y) = |y_1| + |y_2| + \sqrt{2}|y_3|, \quad w_2(Y) = \sqrt{2}(|y_1| + |y_3|), \quad w(Y) = \max(w_1(Y), w_2(Y)).$$

For a lattice centre $p = (x_j, y, z_m)$ and $Y = p/D$ in a fixed bounded set, the elementary hyperbolic-function estimates give, uniformly,

$$\frac{1}{D} \log |\gamma_{jm}| = \sqrt{2}(|y_1| - |y_3| - \sigma) + O(D^{-1}). \quad (25)$$

The right side of the stability criterion (23) has rate

$$\frac{1}{D} \log \left(\frac{1}{50} \frac{\lambda_D^2 \cosh^2(\sqrt{2}x_j)}{\cosh(m\pi)} \right) = \sqrt{2}(2|y_1| - |y_3| - 2\sigma) + O(D^{-1}). \quad (26)$$

By Lemma 3.5, the logarithmic upper rate of the left side of (23) is at most

$$\max(\varphi(w_1(Y)), \varphi(w_2(Y)) - \sqrt{2}\sigma) \leq \varphi(w(Y)).$$

The last inequality uses $\sigma \geq 0$ and monotonicity of φ . Accordingly, the limiting region whose strict interior supplies stable lattice centres is

$$\begin{aligned} V(\sigma) &= \left\{ Y : |y_1| \leq |y_3| + \sigma, \varphi(w(Y)) \leq \sqrt{2}(2|y_1| - |y_3| - 2\sigma) \right\}, \\ w(Y) &= \max(|y_1| + |y_2| + \sqrt{2}|y_3|, \sqrt{2}(|y_1| + |y_3|)). \end{aligned} \quad (27)$$

Thus (27) is deliberately conservative only in discarding the nonpositive contribution $-\sqrt{2}\sigma$ from the second Taylor tail.

Theorem 3.6. *For every $\sigma \geq 0$,*

$$R(D) \geq (\rho \cdot \text{vol } V(\sigma) - o(1))D^3, \quad \rho = \frac{\sqrt{2}}{\pi^3},$$

as $D \rightarrow \infty$, and $R(D) \leq (D-1)^3$ by affine Bézout.

Proof. We first justify the asymptotic count. The elementary argument in Lemma 4.1 below shows, independently of the present theorem, that $w < 1$ throughout $V(\sigma)$, so $V(\sigma)$ is bounded. We also record why its boundary has measure zero. Write

$$a = |y_1|, \quad b = |y_2|, \quad c = |y_3|, \quad q_\sigma(a, c) = \exp(\sqrt{2}(2a - c - 2\sigma) - 1).$$

By the same lemma, on each of the eight orthants the region is described by

$$\begin{aligned} a, b, c &\geq 0, & a - c - \sigma &\leq 0, \\ a + b + \sqrt{2}c - q_\sigma(a, c) &\leq 0, & \sqrt{2}(a + c) - q_\sigma(a, c) &\leq 0, & q_\sigma(a, c) &< 1. \end{aligned}$$

A boundary point either lies on a coordinate plane or makes one of the four non-coordinate constraints in the display an equality. All are among finitely many zero sets of nonzero real-analytic functions. Each such zero set has three-dimensional Lebesgue measure zero; hence $\partial V(\sigma)$ is null.

For the lower bound, there is nothing to prove if $\text{vol } V(\sigma) = 0$, so assume that $\text{vol } V(\sigma) > 0$ and fix $\varepsilon > 0$. The subset on which both inequalities in (27) are strict has full measure in $V(\sigma)$: the omitted equality sets are among the null real-analytic zero sets just considered. We may therefore choose a finite union $\mathcal{K}$ of pairwise disjoint closed axis-parallel boxes, contained in this strict subset, disjoint from a neighbourhood of the origin, and satisfying $\text{vol}(\mathcal{K}) > \text{vol } V(\sigma) - \varepsilon$. Indeed, approximate the open strict subset from within by finitely many boxes, shrink the boxes slightly to make their closures disjoint, and remove a ball about the origin of arbitrarily small volume. Compactness and the strict inequalities give an $\eta > 0$ such that on $\mathcal{K}$

$$|y_1| \leq |y_3| + \sigma - \eta, \quad \varphi(w(Y)) \leq \sqrt{2}(2|y_1| - |y_3| - 2\sigma) - \eta. \quad (28)$$

We first show that the relevant index pairs are admissible, *before* invoking Proposition 3.2; this ordering matters, because γ_{jm} must be known to lie in $[-1, 1]$ before any critical point of the form (x_j, y, z_m) exists at all. Let $I_1 \times I_2 \times I_3$ be one of the boxes constituting $\mathcal{K}$, and let (j, m) be an index pair with $x_j/D \in I_1$ and $z_m/D \in I_3$. The *first* inequality in (28) involves only y_1 and y_3 , hence only x_j and z_m , and may therefore be applied to such a pair with no reference to y . Combining it with

$$\frac{1}{D} \log |\gamma_{jm}| = \sqrt{2} \left(\frac{|x_j|}{D} - \frac{|z_m|}{D} - \sigma \right) + O(D^{-1}),$$

whose error is uniform because $\log |\sinh t| = |t| + O(1)$ for $|t| \geq \pi/\sqrt{2}$ and $\log \cosh t = |t| + O(1)$ on $\mathbb{R}$, we obtain

$$|\gamma_{jm}| \leq \frac{1}{2}$$

for all sufficiently large D , uniformly over all such index pairs. In particular $|\gamma_{jm}| < 1$, so Proposition 3.2 does apply: the two y -sequences $\pm \arccos(\gamma_{jm}) + 2\pi\mathbb{Z}$ are well defined, and every point $p = (x_j, y, z_m)$ with y in one of them is a nondegenerate critical point of Φ_{λ_D} lying in the selected set of Section 3.2. From here on we consider those points p that additionally satisfy $y/D \in I_2$, so that $p/D \in \mathcal{K}$.

To verify the Taylor criterion uniformly, put $u_i = W_i(p)/D = w_i(p/D)$ for $i = 1, 2$. Since $\mathcal{K}$ avoids the origin, w has a positive minimum on $\mathcal{K}$. Thus, uniformly for $p/D \in \mathcal{K}$ and also along sequences for which $u_i \rightarrow 0$,

$$\varphi(u_i + 0.043/D) \leq \varphi(w(p/D) + 0.043/D) = \varphi(w(p/D)) + o(1).$$

Applying (24) and using this estimate gives, uniformly on $\mathcal{K}$,

$$\begin{aligned} \frac{1}{D} \log(2S_D(W_1(p) + 0.043) + 2\lambda_D S_D(W_2(p) + 0.043)) &\leq \varphi(w(p/D)) + o(1), \\ \frac{1}{D} \log \left(\frac{1}{50} \frac{\lambda_D^2 \cosh^2(\sqrt{2}x_j)}{\cosh(m\pi)} \right) &= \sqrt{2} \left(\frac{2|x_j|}{D} - \frac{|z_m|}{D} - 2\sigma \right) + o(1). \end{aligned}$$

The second margin in (28) therefore implies (23) for all sufficiently large D . Every such point contributes a ball of nonzero local degree, and the balls are pairwise disjoint by Section 3.2.

It remains to count the centres in $D\mathcal{K}$. The x -coordinates have spacing π , the z -coordinates have spacing $\pi/\sqrt{2}$, and for each admissible pair (j, m) as above — for which $|\gamma_{jm}| \leq 1/2$ has already been established, so that $\arccos(\gamma_{jm})$ is defined — the two y -sequences are $\pm \arccos(\gamma_{jm}) + 2\pi\mathbb{Z}$. Uniformly in their shifts, the number of their points in the dilated interval DI_2 is

$$\frac{D|I_2|}{\pi} + O(1).$$

Hence a box $I_1 \times I_2 \times I_3$, after dilation by D , contains

$$\frac{\sqrt{2}}{\pi^3} |I_1| |I_2| |I_3| D^3 + O(D^2)$$

centres. Summing over the finitely many disjoint boxes gives $\rho \operatorname{vol}(\mathcal{K}) D^3 + O(D^2)$. For this fixed $\mathcal{K}$, the preceding count supplies $\rho \operatorname{vol}(\mathcal{K}) D^3 + O(D^2)$ pairwise disjoint balls of nonzero local degree. We now make the polynomial Morse without losing these balls.

For fixed D there are finitely many selected balls, and the union S of their boundaries is compact. On S the gradient of $T_D \Phi_{\lambda_D}$ is bounded away from zero: by Lemma 2.3 applied to Φ_{λ_D} , $|\nabla \Phi_{\lambda_D}| \geq \delta_p$ on $\partial B(p, r_p)$, while the criterion (23) bounds $|\nabla E| < \delta_p$ there, so

$$|\nabla T_D \Phi_{\lambda_D}| \geq |\nabla \Phi_{\lambda_D}| - |\nabla E| > 0 \quad \text{on } S,$$

and $\mu := \min_S |\nabla T_D \Phi_{\lambda_D}| > 0$ by compactness. If Q is a harmonic polynomial of degree at most D with $\sup_S |\nabla Q| < \mu$, then $\nabla(T_D \Phi_{\lambda_D} + Q)$ does not vanish on S and is homotopic to $\nabla T_D \Phi_{\lambda_D}$ through nonvanishing fields there; hence every boundary degree is preserved. Since the space of harmonic polynomials of degree at most D is finite dimensional, all norms on it are equivalent and this is an open condition on the coefficients of Q .

The function Φ_{λ_D} is even in each variable, so if D is odd, $T_D \Phi_{\lambda_D} = T_{D-1} \Phi_{\lambda_D}$. In this case first add a sufficiently small nonzero harmonic homogeneous polynomial of degree D , for example a scalar multiple of $\operatorname{Re}((x + iy)^D)$, viewed as a polynomial independent of z . If D is even, the coefficient of y^D in $T_D \Phi_{\lambda_D}$ is $(-1)^{D/2}/D! \neq 0$. Thus in either case we have, without changing any selected boundary degree, a harmonic polynomial P_D^0 with nonzero degree- D part. Choose an open coefficient neighbourhood $\mathcal{U}_D$ of P_D^0 small enough that every selected boundary degree is preserved and every polynomial in $\mathcal{U}_D$ has a nonzero degree- D part.

Let $\mathcal{P}_D^{\text{harm}}$ denote the space of real harmonic polynomials of degree at most D . The universal evaluation map

$$\mathcal{F} : \mathcal{P}_D^{\text{harm}} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \mathcal{F}(P, x) = \nabla P(x),$$

is a submersion: varying the three linear coefficients changes its value by an arbitrary vector in $\mathbb{R}^3$. Parametric transversality [8] therefore supplies a polynomial in $\mathcal{U}_D$ for which ∇P is transverse to 0. Every real critical point is then nondegenerate, and all selected boundary degrees remain nonzero. The real critical set is both semialgebraic and discrete; since a semialgebraic set has only finitely many connected components, this set is finite. Each selected ball contains a distinct nondegenerate critical point, proving the lower bound. Consequently, for every fixed $\mathcal{K}$ as above,

$$\liminf_{D \rightarrow \infty} \frac{R(D)}{D^3} \geq \rho \operatorname{vol}(\mathcal{K}).$$

Letting $\varepsilon \downarrow 0$ proves the asserted lower bound.

Finally, for the upper bound, let P be any real harmonic polynomial of degree at most D . If $D \geq 2$, the three components of ∇P have degree at most $D - 1$. Every nondegenerate real critical point is an isolated nonsingular solution of the same equations over $\mathbb{C}$. The affine Bézout inequality for isolated solutions therefore bounds their number by $(D - 1)^3$. This form of the inequality remains valid when the common complex zero set has positive-dimensional components. Indeed, take any finite collection of nonsingular isolated zeros. A sufficiently small simultaneous perturbation of the three equations preserves a zero near every member of this collection. Within every such coefficient neighbourhood, the systems whose degree- $(D - 1)$ homogenizations meet properly in projective space form a dense Zariski-open set. The usual projective Bézout theorem

then bounds the size of the collection by $(D - 1)^3$. Since the collection was arbitrary, the same bound holds for all nonsingular isolated zeros of the original system. If $D = 1$, the gradient is constant, so there are no nondegenerate critical points. Taking the supremum over P proves the upper bound for $R(D)$. $\square$

4 The volume in closed form, and a certified constant

To make Theorem 3.6 quantitative, we first reduce $\text{vol } V(\sigma)$ to an explicit two-dimensional integral and then certify a lower bound for that integral.

Throughout write $a = |y_1|$, $b = |y_2|$, $c = |y_3|$ and

$$\mathcal{R}(a, c) = \exp(\sqrt{2}(2a - c - 2\sigma) - 1). \quad (29)$$

Lemma 4.1 (No exponential branch). *On $V(\sigma)$ one has $w(Y) < 1$ and $\mathcal{R}(a, c) < 1$. Consequently, membership in $V(\sigma)$ is equivalent to*

$$a \leq c + \sigma, \quad w(Y) \leq \mathcal{R}(a, c) < 1. \quad (30)$$

Proof. Always $w \geq \sqrt{2}(a + c)$. Suppose $w \geq 1$; then $\varphi(w) = w$, so the defining inequality gives $\sqrt{2}(a + c) \leq w \leq \sqrt{2}(2a - c - 2\sigma)$, i.e. $a + c \leq 2a - c - 2\sigma$, i.e. $2c + 2\sigma \leq a$. Combined with $a \leq c + \sigma$ this yields $2c + 2\sigma \leq c + \sigma$, i.e. $c + \sigma \leq 0$. For $\sigma > 0$ this is impossible; for $\sigma = 0$ it forces $c = 0$ and then $a = 0$. But the defining inequality would then give $w \leq 0$, contradicting $w \geq 1$. Hence $w < 1$ on $V(\sigma)$. If $w > 0$, the second defining inequality is therefore equivalent to

$$1 + \log w \leq \sqrt{2}(2a - c - 2\sigma), \quad \text{or equivalently} \quad w \leq \mathcal{R}(a, c).$$

If $w = 0$, both the defining inequality and $w \leq \mathcal{R}(a, c)$ hold because $\varphi(0) = -\infty$ and $\mathcal{R}(a, c) > 0$. Moreover, $a \leq c + \sigma$ gives

$$\sqrt{2}(2a - c - 2\sigma) \leq \sqrt{2}c \leq w < 1,$$

so the exponent in (29) is negative and $\mathcal{R} < 1$. Conversely, (30) implies $w < 1$; reversing the preceding equivalence when $w > 0$, and using the convention $\varphi(0) = -\infty$ when $w = 0$, recovers the two defining inequalities of $V(\sigma)$. $\square$

Lemma 4.2 (Closed form). *For every $\sigma \geq 0$,*

$$\text{vol } V(\sigma) = 8 I(\sigma), \quad I(\sigma) = \iint_{\Omega(\sigma)} (\mathcal{R}(a, c) - a - \sqrt{2}c) da dc, \quad (31)$$

where

$$\Omega(\sigma) = \{(a, c) : 0 \leq a \leq c + \sigma, \quad 0 \leq c, \quad \sqrt{2}(a + c) \leq \mathcal{R}(a, c) < 1\},$$

and the integrand is nonnegative on $\Omega(\sigma)$.

Proof. $V(\sigma)$ is invariant under each sign flip $y_i \mapsto -y_i$. The three coordinate planes have measure zero, so its volume is eight times its volume in the positive octant, where $a, b, c \geq 0$ by definition. By Lemma 4.1 the condition there is $w \leq \mathcal{R} < 1$, i.e. both $\sqrt{2}(a + c) \leq \mathcal{R}$ and $a + b + \sqrt{2}c \leq \mathcal{R}$. The first does not involve b ; together with $a \leq c + \sigma$, with $c \geq 0$, and with $\mathcal{R} < 1$ it defines $\Omega(\sigma)$. The second gives the b -extent $0 \leq b \leq \mathcal{R} - a - \sqrt{2}c$. That extent is nonnegative on $\Omega(\sigma)$, since $\sqrt{2}(a + c) \leq \mathcal{R}$ implies $\mathcal{R} - a - \sqrt{2}c \geq \sqrt{2}a - a = (\sqrt{2} - 1)a \geq 0$. Integrating this fibre length in (a, c) and applying Tonelli's theorem gives the formula (31). $\square$

Lemma 4.3 (Exact inner integral). *For fixed a and $0 \leq c_1 \leq c_2$,*

$$\int_{c_1}^{c_2} (\mathcal{R}(a, c) - a - \sqrt{2}c) dc = \frac{\mathcal{R}(a, c_1) - \mathcal{R}(a, c_2)}{\sqrt{2}} - a(c_2 - c_1) - \frac{c_2^2 - c_1^2}{\sqrt{2}}. \quad (32)$$

Proof. $\mathcal{R}(a, c) = \kappa(a)e^{-\sqrt{2}c}$ with $\kappa(a) = \exp(\sqrt{2}(2a - 2\sigma) - 1)$, so $\int \mathcal{R} dc = -\mathcal{R}/\sqrt{2}$; and $\int (a + \sqrt{2}c) dc = ac + \frac{\sqrt{2}}{2}c^2 = ac + c^2/\sqrt{2}$. $\square$

Proposition 4.4 (Positivity, by hand). $\text{vol } V(0) > 0$. Hence $R(D) = \Theta(D^3)$ with no computation.

Proof. Take $\sigma = 0$ and

$$(y_1, y_2, y_3) = \left(\frac{1}{10}, 0, \frac{11}{100} \right).$$

Here the second branch in the maximum defining w is larger by $(\sqrt{2} - 1)/10$, so

$$w = \frac{21\sqrt{2}}{100} < \frac{1}{3} < e^{-1}, \quad \varphi(w) < 0 < \frac{9\sqrt{2}}{100}.$$

(For example, the two elementary inequalities follow from $63\sqrt{2} < 100$ and $e < 3$.) Thus the second defining inequality is strict, while $1/10 < 11/100$ makes the first one strict. By continuity the strict inequalities continue to hold on a neighbourhood of this point. Hence $V(0)$ contains a nonempty open set and has positive volume. Theorem 3.6 then gives $R(D) = \Omega(D^3)$, while its affine Bézout bound gives $R(D) = O(D^3)$. $\square$

Proposition 4.5 (Certified lower bound).

$$\rho \cdot \text{vol } V(37/500) > 1.0172 \times 10^{-3}, \quad \rho \cdot \text{vol } V(0) > 8.139 \times 10^{-4}.$$

Proof. By Lemma 4.2, $\rho \cdot \text{vol } V(\sigma) = \frac{8\sqrt{2}}{\pi^3} I(\sigma)$, so it suffices to bound $I(\sigma)$ below. Partition $[0, 3/5]$ into N exact rational cells $[a_1, a_2]$ of equal length and, for each, set

$$c_{\text{lo}} = \max(0, a_2 - \sigma),$$

which guarantees $a \leq c + \sigma$ for every $a \in [a_1, a_2]$ and $c \geq c_{\text{lo}}$. Every retained rectangle also satisfies $\mathcal{R} < 1$. Indeed, if $a_2 \geq \sigma$, then throughout the rectangle

$$\mathcal{R}(a, c) \leq \mathcal{R}(a_2, a_2 - \sigma) = \exp(\sqrt{2}(a_2 - \sigma) - 1) \leq \exp(3\sqrt{2}/5 - 1) < 1;$$

if $a_2 < \sigma$, then $\mathcal{R}(a, c) \leq \mathcal{R}(a_2, 0) < e^{-1} < 1$. Here $3\sqrt{2} < 5$. Since $\mathcal{R}$ is increasing in a and decreasing in c , the inequality

$$\sqrt{2}(a_2 + c_{\text{hi}}) \leq \mathcal{R}(a_1, c_{\text{hi}}) \quad (33)$$

implies $\sqrt{2}(a + c) \leq \mathcal{R}(a, c)$ for every $(a, c) \in [a_1, a_2] \times [c_{\text{lo}}, c_{\text{hi}}]$; we take c_{hi} to be any value for which (33) is verified, obtained by bisection on the strictly decreasing function $c \mapsto \mathcal{R}(a_1, c) - \sqrt{2}(a_2 + c)$, whose derivative is $-\sqrt{2}\mathcal{R}(a_1, c) - \sqrt{2} < 0$. If the inequality does not hold strictly at c_{lo} , the cell is omitted; otherwise bisection in the initial interval $[c_{\text{lo}}, c_{\text{lo}} + 2]$ returns a certified directed point $c_{\text{hi}} \geq c_{\text{lo}}$ satisfying the condition (33); cells returning $c_{\text{hi}} = c_{\text{lo}}$ are discarded, which is legitimate because the integrand is nonnegative on $\Omega(\sigma)$. That interval is a valid bracket: at its right endpoint $c \geq 2$, so, using $a_1 \leq 3/5$,

$$\mathcal{R}(a_1, c) \leq \exp(\sqrt{2}(6/5 - 2) - 1) < \frac{1}{6} < 2\sqrt{2} \leq \sqrt{2}(a_2 + c),$$

where the first strict inequality follows from $\sqrt{2} > 5/4$ and $e^2 > 6$. Thus the bisected function is negative there. Using again monotonicity of $\mathcal{R}$, on the resulting rectangle

$$\mathcal{R}(a, c) - a - \sqrt{2}c \geq \mathcal{R}(a_1, c) - a_2 - \sqrt{2}c,$$

whose c -integral over $[c_{\text{lo}}, c_{\text{hi}}]$ is given exactly by formula (32), with a replaced by a_2 in the linear terms and a_1 in $\mathcal{R}$. For $c \leq c_{\text{hi}}$, the cell condition also gives

$$\mathcal{R}(a_1, c) - a_2 - \sqrt{2}c \geq \sqrt{2}(a_2 + c) - a_2 - \sqrt{2}c = (\sqrt{2} - 1)a_2 \geq 0.$$

Summing the resulting lower bounds times the cell widths gives a lower bound for $I(\sigma)$, because the cells are disjoint subsets of $\Omega(\sigma)$ up to their measure-zero common boundaries, and the integrand is nonnegative there (Lemma 4.2); any nonpositive cell bound may therefore be replaced by zero.

The accompanying script implements this construction using Arb ball arithmetic [9] through `python-flint 0.9.0`, at 200-bit precision. It uses the exact rational inputs $\sigma \in \{0, 37/500\}$, $a_{\text{max}} = 3/5$, and $N = 120000$. Lower and upper enclosures of each exact cell endpoint are selected according to monotonicity: the left endpoint is rounded down where it occurs in a lower bound for $\mathcal{R}$, and the right endpoint is rounded up where it occurs in an increasing bound. The underlying rational cells remain disjoint; rounding enlarges only these worst-case estimates. In particular, c_{lo} uses a lower enclosure of σ . The cell condition and integrand use an upper enclosure of σ , producing lower bounds for $\mathcal{R}$; the separate verification of $\mathcal{R} < 1$ uses a lower enclosure of σ , producing an upper bound for $\mathcal{R}$. The bisection performs 90 steps and returns a directed point endpoint that is rechecked against (33). The lower endpoint of each cell width is used. Each antiderivative is evaluated as an enclosing ball, after which only its certified lower endpoint is accumulated. Finally, the program compares the accumulated lower-bound balls directly with exact rational thresholds; decimal conversion is not used in any logical decision. It certifies

$$I(37/500) > 2.7881 \times 10^{-3}, \quad I(0) > 2.23066 \times 10^{-3},$$

and, by direct exact-threshold tests,

$$\frac{8\sqrt{2}}{\pi^3} I(37/500) > 1.0172 \times 10^{-3}, \quad \frac{8\sqrt{2}}{\pi^3} I(0) > 8.139 \times 10^{-4}.$$

The nonauthoritative decimal diagnostics for the two latter quantities are $1.01736092 \times 10^{-3}$ and $8.14001610 \times 10^{-4}$, respectively. The script is `certificates/certify.py`; its sole external dependency is pinned in `certificates/requirements.txt`. $\square$

Remark 4.6. The truncation of the a -range to $[0, 3/5]$ loses nothing for either certified value of σ . Indeed, if $(a, c) \in \Omega(\sigma)$, then $c \geq a - \sigma$, while Lemma 4.1 gives $\sqrt{2}(a + c) \leq w < 1$. Therefore

$$\sqrt{2}(2a - \sigma) \leq \sqrt{2}(a + c) < 1, \quad a < \frac{1/\sqrt{2} + \sigma}{2}.$$

For $\sigma \in \{0, 37/500\}$ the last upper bound is less than $3/5$.

Corollary 4.7. *As $D \rightarrow \infty$, $R(D) \geq (1.0172 \times 10^{-3} - o(1))D^3$; moreover, $R(D) \leq (D - 1)^3$ for every $D \geq 1$. In particular, for every $c < 1.0172 \times 10^{-3}$ one has $R(D) \geq cD^3$ for all sufficiently large D .*

Proof. Apply Theorem 3.6 with $\sigma = 37/500$ and use Proposition 4.5; the upper bound is the second assertion of that theorem. The final statement follows because the $o(1)$ term tends to zero. $\square$

5 $N(D) \leq (D+1)(2D+1)$

Definition 5.1. For $D \geq 1$, let $N(D)$ be the least integer N with the following property: for every nonzero $h_D \in \mathcal{H}_D$, there are distinct nodes $b_1, \dots, b_N \in \mathbb{R}^3 \setminus \{0\}$ and a vector $w^* \in \mathbb{R}_{>0}^N$ for which the associated map A in Lemma 2.5 is surjective and $A(w^*) = (0, \dots, 0, h_D)$. Theorem 5.2 below proves that such integers exist. Surjectivity also gives the immediate lower bound $N(D) \geq \dim W = D^2 + 2D$.

Theorem 5.2. Let $D \geq 1$ and $0 \neq h_D \in \mathcal{H}_D$. There are $n_0 = (D+1)(2D+1)$ distinct points $b_i \in \mathbb{S}^2$ and weights $w_i^* > 0$ satisfying (R1) and (R2) of Lemma 2.5, with degree- D component equal to the prescribed polynomial h_D . The points may be chosen independently of h_D . Consequently $N(D) \leq (D+1)(2D+1)$.

Proof. Step 1 (a positive density). Let $d\sigma$ be normalized surface measure on $\mathbb{S}^2$. Symmetry of the zonal harmonic and (10) give

$$(2l+1) \int_{\mathbb{S}^2} Z_l(b; x) h(b) d\sigma(b) = h(x), \quad h \in \mathcal{H}_l, \quad x \in \mathbb{R}^3.$$

Choose

$$c > (2D+1) \max_{b \in \mathbb{S}^2} |h_D(b)|, \quad g(b) = c + (2D+1)h_D(b) \quad (b \in \mathbb{S}^2),$$

so g is strictly positive. For fixed x , the function $b \mapsto Z_l(b; x)$ belongs to $\mathcal{H}_l$. Its integral is zero for $l \geq 1$ by the mean value property. Orthogonality of distinct spherical-harmonic degrees and the displayed reproducing identity therefore give, for $1 \leq l \leq D$,

$$\int_{\mathbb{S}^2} g(b) Z_l(b; x) d\sigma(b) = \begin{cases} 0, & l < D, \\ h_D(x), & l = D. \end{cases} \quad (34)$$

Step 2 (an exact positive cubature). Put $K = D+1$ and $M = 2D+1$. Let $u_1, \dots, u_K \in (-1, 1)$ and $v_1, \dots, v_K > 0$ be the Gauss–Legendre nodes and weights, normalized so that

$$\int_{-1}^1 p(u) du = \sum_{k=1}^K v_k p(u_k) \quad (\deg p \leq 2K-1).$$

For $0 \leq j < M$, set $\phi_j = 2\pi j/M$ and

$$b_{k,j} = \left(\sqrt{1-u_k^2} \cos \phi_j, \sqrt{1-u_k^2} \sin \phi_j, u_k \right), \quad W_{k,j} = \frac{v_k}{2M}.$$

These $KM = (D+1)(2D+1)$ nodes are distinct, because the Gauss nodes are distinct and lie strictly between the poles, and all their weights are positive. We claim that

$$\int_{\mathbb{S}^2} p(b) d\sigma(b) = \sum_{k=1}^K \sum_{j=0}^{M-1} W_{k,j} p(b_{k,j}) \quad (\deg p \leq 2D). \quad (35)$$

Indeed,

$$\frac{1}{M} \sum_{j=0}^{M-1} e^{im\phi_j} = \begin{cases} 1, & m = 0, \\ 0, & 0 < |m| \leq 2D, \end{cases}$$

so the discrete azimuthal average agrees with the continuous one for every trigonometric polynomial of degree at most $2D$. By linearity it remains to consider a monomial $x^\alpha y^\beta z^\gamma$ of total degree at most $2D$. Its continuous azimuthal average is zero unless both α and β are even; when they are even, it is

$$C_{\alpha,\beta}(1-u^2)^{(\alpha+\beta)/2}u^\gamma$$

for a constant $C_{\alpha,\beta}$, hence is a polynomial in u of degree $\alpha + \beta + \gamma \leq 2D$. The Gauss–Legendre rule is exact through degree $2K - 1 = 2D + 1$, and $d\sigma = (4\pi)^{-1} d\phi du$, proving the formula (35) and its normalization factor $1/(2M)$.

The function g is the restriction of a polynomial of degree D , while $Z_l(b; x)$ is a polynomial of degree l in b . Thus, for fixed x and $1 \leq l \leq D$, their product has degree at most $D + l \leq 2D$. Equations (35) and (34) give

$$\sum_{k,j} W_{k,j} g(b_{k,j}) Z_l(b_{k,j}; x) = \int_{\mathbb{S}^2} g(b) Z_l(b; x) d\sigma(b).$$

Thus $w_{k,j}^* := W_{k,j} g(b_{k,j}) > 0$ realizes exactly $A(w^*) = (0, \dots, 0, h_D)$, which is (R2).

Step 3 (full row rank). Define the evaluation map

$$E : W \longrightarrow \mathbb{R}^{KM}, \quad E(f_1, \dots, f_D) * k, j = \sum * l = 1^D f_l(b_{k,j}).$$

We show that E is injective. Suppose that $f = \sum_{l=1}^D f_l$ vanishes at every node. The polynomial f^2 has degree at most $2D$, so the exact cubature (35) and positivity of its weights give

$$\int_{\mathbb{S}^2} f(b)^2 d\sigma(b) = \sum_{k,j} W_{k,j} f(b_{k,j})^2 = 0.$$

Hence f vanishes identically on $\mathbb{S}^2$. Orthogonality of the different spherical-harmonic degrees then gives $f_l = 0$ for every l , proving injectivity of E .

To pass from evaluation to the map A in Lemma 2.5, equip each $\mathcal{H}_l$ with its $L^2(\mathbb{S}^2, d\sigma)$ inner product and W with the orthogonal direct-sum inner product. If $e_{k,j}$ denotes the corresponding coordinate vector in $\mathbb{R}^{KM}$, the reproducing identity implies

$$\langle Ae_{k,j}, (f_1, \dots, f_D) \rangle_W = \sum_{l=1}^D \frac{f_l(b_{k,j})}{2l+1}.$$

If $(f_1, \dots, f_D) \in \ker A^*$, the right-hand side vanishes at every node. Injectivity of E , applied to the tuple $(f_l/(2l+1))_{l=1}^D \in W$, forces every f_l to vanish. Thus $\ker A^* = \{0\}$, and finite-dimensional duality gives $\text{im } A = W$. This is (R1). $\square$

Remark 5.3. The product rule is retained because it simultaneously gives positivity, exactness, and the spanning property (R1). Conic Carathéodory can reduce a particular positive representation to at most $\dim W$ nodes. Its surviving columns, however, need not span W ; if fewer than $\dim W$ survive, (R1) necessarily fails. The strict positivity required in (R2) also prevents one from simply restoring discarded nodes with zero weights. No such reduction is used here.

Corollary 5.4. *For every $D \geq 1$,*

$$M(N(D)) \geq R(D), \quad M((D+1)(2D+1)) \geq R(D).$$

Proof. Let P be a harmonic polynomial of degree at most D , and fix any finite collection $p_1, \dots, p_r$ of its nondegenerate real critical points. If $r = 0$, the desired inequality is immediate, so assume $r \geq 1$. If $\deg P < D$, choose any nonzero $h_D \in \mathcal{H}_D$. For all sufficiently small nonzero t , the implicit function theorem continues the p_j to distinct nondegenerate critical points of $P + th_D$. Replacing P by one such perturbation when necessary, and relabelling the continued critical points as $p_1, \dots, p_r$, we may therefore assume that its degree- D part is nonzero while retaining r counted points.

The constant term of P does not affect its critical points. Apply the definition of $N(D)$ and Lemma 2.5 to the homogeneous components of $P - P(0)$. For all sufficiently small $\varepsilon > 0$, write

$$F_\varepsilon(X) = \varepsilon^{-D}(V_\varepsilon(\varepsilon X) - V_\varepsilon(0)).$$

Lemma 2.5 gives $F_\varepsilon \rightarrow P - P(0)$ locally in C^∞ , while the chain rule gives

$$\nabla_X F_\varepsilon(X) = \varepsilon^{1-D} \nabla V_\varepsilon(\varepsilon X).$$

Thus the critical points of F_ε correspond exactly, under $X \mapsto \varepsilon X$, to those of V_ε . Since every p_j is isolated, choose pairwise disjoint closed balls $B_j = \bar{B}(p_j, s_j)$ centred at p_j with $\text{Crit}(P) \cap B_j = \{p_j\}$. Centring matters: hypothesis (P2) of Theorem 2.1 requires that no critical point of the limit lie on ∂K , and with $K = \bigcup_{j=1}^r B_j$ the only critical points of P in K are the centres p_j , which lie in $\text{int } K$. Hypothesis (P1) holds because each p_j is nondegenerate. Apply Theorem 2.1 with $H = P - P(0)$, $F = F_\varepsilon$, and $K = \bigcup_{j=1}^r B_j$; by its localized form, for ε small F_ε has exactly one nondegenerate critical point in each B_j , of the same Morse index as p_j . Undoing the rescaling gives r such points for a potential of $N(D)$ distinct positive charges.

Finally fix the positive strengths and perturb the source locations arbitrarily slightly. The submersion and compactness argument in Proposition 1.3 makes the full potential Morse with a finite critical set, while the r counted points persist. Hence $M(N(D)) \geq r$. Since P and the finite collection were arbitrary, taking the supremum in Definition 1.1 proves $M(N(D)) \geq R(D)$. Theorem 5.2 gives $N(D) \leq (D+1)(2D+1)$, and repeated application of Proposition 1.3 proves the second inequality. $\square$

6 Proof of the main theorem

Proof of Theorem 1.4. Let

$$A_0 = 1.0172 \times 10^{-3}, \quad N_D = (D+1)(2D+1) = 2D^2 + 3D + 1.$$

Define

$$\delta_D := \max\left(0, A_0 - \frac{R(D)}{D^3}\right) \geq 0,$$

so that $R(D) \geq (A_0 - \delta_D)D^3$ holds for *every* $D \geq 1$ by construction, with no case distinction and no ad hoc enlargement of terms. Corollary 4.7 states exactly that $R(D)/D^3 \geq A_0 - o(1)$, i.e. that $\delta_D \rightarrow 0$. Combining this with Corollary 5.4 yields

$$M(N_D) \geq R(D) \geq (A_0 - \delta_D)D^3. \tag{36}$$

For an integer $n \geq 6$, set

$$x_n = \frac{\sqrt{8n+1} - 3}{4}, \quad D_n = \lfloor x_n \rfloor.$$

Then $x_n \geq 1$, and direct solution of the quadratic inequality gives

$$N_D \leq n \iff D \leq x_n \quad (D \in \mathbb{Z}_{\geq 0}).$$

Consequently $N_{D_n} \leq n < N_{D_n+1}$, and repeated application of Proposition 1.3, followed by (36), gives

$$M(n) \geq M(N_{D_n}) \geq (A_0 - \delta_{D_n})D_n^3. \quad (37)$$

Since

$$D_n = x_n + O(1) = \sqrt{n/2} + O(1),$$

the numbers

$$\theta_n = \frac{2\sqrt{2} D_n^3}{n^{3/2}}$$

tend to 1. They also satisfy $0 < \theta_n \leq 1$: positivity follows from $n \geq 6$, while $N_{D_n} \leq n$ and $N_D \geq 2D^2$ give $D_n \leq \sqrt{n/2}$. Define

$$\epsilon_n = \frac{A_0(1 - \theta_n) + \delta_{D_n}\theta_n}{2\sqrt{2}}.$$

Since $D_n \rightarrow \infty$, one has $\delta_{D_n} \rightarrow 0$; hence $\epsilon_n \geq 0$ and $\epsilon_n \rightarrow 0$. Rewriting (37) now gives

$$M(n) \geq \left(\frac{A_0}{2\sqrt{2}} - \epsilon_n \right) n^{3/2} = (3.596345089 \dots \times 10^{-4} - \epsilon_n) n^{3/2}.$$

This proves both the more precise constant and the boxed coefficient in Theorem 1.4, with the asserted nonnegative error term. For $2 \leq n < 6$, setting $\epsilon_n = c_0$ extends the displayed inequality to every n in the domain of M without affecting the limit.

Finally, Proposition 4.4 and Theorem 3.6 with $\sigma = 0$ give $R(D) \geq cD^3$ for some $c > 0$ and all sufficiently large D , without numerical computation. Corollary 5.4 and the same choice of D_n then give $M(n) = \Omega(n^{3/2})$ with a computation-free positive implicit constant. $\square$

Remark 6.1 (The $\lambda_D = 1$ variant). Taking $\sigma = 0$, so that $\lambda_D = 1$, and using the certified estimate $\rho \operatorname{vol} V(0) > 8.139 \times 10^{-4}$ from Proposition 4.5, the same argument gives

$$M(n) \geq (2.877571046 \dots \times 10^{-4} - o(1)) n^{3/2}.$$

Here the $o(1)$ may again be taken nonnegative. This value uses the computer-assisted certificate, but it requires no optimization over σ . The qualitative positive constant in the final paragraph of the proof uses only Proposition 4.4.

7 Scope of the result

7.1 Interpretation of the numerical constant

Write

$$c_0 = \frac{1.0172 \times 10^{-3}}{2\sqrt{2}} = 3.596345089 \dots \times 10^{-4}.$$

Theorem 1.4 gives a particular sequence $\epsilon_n \geq 0$, with $\epsilon_n \rightarrow 0$, for which $M(n) \geq (c_0 - \epsilon_n) n^{3/2}$. If the error term is suppressed solely to compare scales, then

$$c_0 n^{3/2} = kn \iff n = \left(\frac{k}{c_0} \right)^2.$$

Thus the leading term equals n at approximately 7.73×10^6 and equals $10n$ at approximately 7.73×10^8 . These numbers are not finite- n thresholds: the proof gives no effective rate for $\epsilon_n \rightarrow 0$.

For comparison, Proposition 1.2 gives $\# \text{Crit}(V) = 2m_1 + n - 1 \geq n - 1$ for every potential admitted in the definition of $M(n)$, and the final paragraph of Proposition 1.3 shows that this class is nonempty. Hence $M(n) \geq n - 1$ for every $n \geq 2$. More strongly, Arathoon, Ball, and Kvalheim [2, Proposition 1] construct, for every $m \geq 0$, a $(3 + 2m)$ -charge potential whose critical set is finite, has cardinality at least $4 + 20m$, and consists entirely of nondegenerate points. Together with Proposition 1.3, this gives the explicit bounds

$$M(n) \geq \begin{cases} 10n - 26, & n \geq 3 \text{ odd}, \\ 10n - 36, & n \geq 4 \text{ even}. \end{cases}$$

These linear estimates, unlike the bound in Theorem 1.4, are explicit at each indicated n . The calculation above only locates the formal leading-term crossover; it asserts no numerical dominance at those values.

7.2 Limits of our method

Theorem 1.4 does not by itself furnish further violations of the quadratic Maxwell benchmark, since $n^{3/2} = o(n^2)$. To state the obstruction within the present two-stage method without conflating the extremal quantities $R(D)$ and $N(D)$, let $r(D)$ denote the number of critical points retained from a chosen degree- D harmonic polynomial and let $s(D)$ denote the number of sources used to realize it. Suppose, hypothetically, that

$$r(D) \sim AD^3, \quad s(D) \sim BD^p, \quad A, B > 0, \quad p > 0.$$

Since $s(D) \rightarrow \infty$, comparison with the literal benchmark $(s(D) - 1)^2$ gives

$$\frac{r(D)}{(s(D) - 1)^2} \sim \frac{A}{B^2} D^{3-2p}. \quad (38)$$

Consequently, $p < 3/2$ forces this ratio to infinity, whereas $p > 3/2$ forces it to zero. If $p = 3/2$, then $A > B^2$ gives an eventual strict improvement and $A < B^2$ makes one impossible; the boundary case $A = B^2$ is not decided by leading asymptotics. The affine Bézout bound also forces $A \leq 1$: writing $r(D)$ for the number of critical points actually produced by a local model of degree D , one has $r(D) \leq R(D) \leq (D - 1)^3$, so $A = \lim r(D)/D^3 \leq 1$. Consequently a strict leading-order improvement at the boundary exponent would in particular require $B < 1$.

For the explicit cubature of Theorem 5.2, $s(D) = N_D = (D + 1)(2D + 1)$. The affine Bézout bound $r(D) \leq R(D) \leq (D - 1)^3$ therefore gives the exact estimate

$$\frac{r(D)}{(N_D - 1)^2} \leq \frac{(D - 1)^3}{D^2(2D + 3)^2} \sim \frac{1}{4D}.$$

Thus no improvement of the packet constant, even up to the Bézout ceiling, can overcome this explicit quadratic realization cost. More generally, under the power-law hypothesis above, equation (38) shows that a leading-order improvement over the quadratic benchmark would require $s(D) = O(D^{3/2})$; the condition $s(D) = o(D^{3/2})$ is sufficient, while at the boundary scale the leading constants and, in the equality case, lower-order terms matter.

The rank obstruction in the present realization lemma is stronger than a property of the particular product cubature. A realization using $s(D)$ sources and satisfying (R1) requires the

linear map $A : \mathbb{R}^{s(D)} \rightarrow W$ to be surjective. Hence $s(D) \geq \text{rank } A = \dim W = D^2 + 2D$, whether or not the nodes were selected as a function of the top homogeneous term h_D . In particular, every realization retaining (R1) obeys

$$\frac{r(D)}{(s(D) - 1)^2} \leq \frac{(D - 1)^3}{(D^2 + 2D - 1)^2} \sim \frac{1}{D}.$$

Therefore a subquadratic source count cannot retain full surjectivity onto all harmonic jets by varying only the weights. A lower-cost transfer would have to abandon that requirement, for example by realizing only a structured family of harmonic polynomials already known to carry $\Theta(D^3)$ critical points, or by using a different target-dependent mechanism. The rank argument does not exclude either possibility.

7.3 Role of computation and reproducibility

The exact certificates and a README file are in the directory `certificates/` in the public github repository github.com/shivakintali/Maxwell-Superlinear

Exactly one logical step in the proof is computer-assisted: the explicit lower bounds in Proposition 4.5. As detailed in its proof, the script uses Arb ball arithmetic through `python-flint` 0.9.0 at 200-bit precision. Its rational parameters and final thresholds are entered exactly; all branch decisions and pass-fail comparisons use Arb enclosures. Conversions to ordinary floating point occur only when printing nonauthoritative decimal diagnostics. With the pinned dependency in `certificates/requirements.txt` installed, the certificate is reproduced by running `python3 certificates/certify.py` from the manuscript directory. The expected pass transcript and SHA-256 checksums are recorded in `certificates/README.md`.

Proposition 4.4 proves $\text{vol } V(0) > 0$ without computation. Together with Theorem 3.6, this proves $R(D) = \Theta(D^3)$; the argument in Section 6 then gives $M(n) = \Omega(n^{3/2})$ with a positive, computation-free implicit constant. Thus ball arithmetic is used only for the displayed numerical coefficient, not for the exponent. The certificate source, requirements file, and reproduction README are intended to accompany the manuscript as supplementary material.

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