More than 42.1% of the Zeros of the Riemann Zeta Function Are on the Critical Line

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Abstract

Let $N_0(T)$ count, with multiplicity, the nontrivial zeros of the Riemann zeta function on $\Re s = \frac{1}{2}$, and let $N(T)$ count all nontrivial zeros. We develop a computer-assisted Levinson–Conrey argument proving

$$\liminf_{T\to\infty} \frac{N_0(T)}{N(T)} > 0.421.$$

The analytic part uses a marked-subset selector on the signed $(-1)^3(\mu * \Lambda^{*3})$ component of a Feng mollifier of limiting length $T^{29/50}$. The finite part uses exact rational polyhedral geometry and rational Taylor remainders. The completed outward certificate gives

$$c_{\mathrm{sel}} \le 2.122496896689520 < 2.122723640072098 \le e^{7527/10000},$$

which closes the strict Levinson–Conrey inequality without an unproved hypothesis.

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Contents

1 Statement, context, and proof architecture

For $T \geq 2$, let $N(T)$ be the number of nontrivial zeros $\rho = \beta + i\gamma$, counted with multiplicity, with $0 < \gamma \leq T$, and let $N_0(T)$ count those with $\beta = \frac{1}{2}$. Put

$$\kappa = \liminf_{T\to\infty} \frac{N_0(T)}{N(T)}.$$

Theorem 1.1. *Unconditionally,*

$$\kappa > \frac{421}{1000}.$$

In particular, more than 42.1% of the nontrivial zeros of $\zeta(s)$ lie on the critical line.

1.1 Context

Levinson [7] proved that more than one third of the zeros are on the critical line, and Conrey [2] raised the proportion above two fifths using the Deshouillers–Iwaniec bilinear estimate [4, 5]. Feng [6] introduced higher convolution pieces, and Pratt, Robles, Zaharescu, and Zeindler [8] obtained $\kappa > 0.417293$ (and hence, in particular, more than five-twelfths). The construction below retains precisely those labelled degree-three tuples having a fixed admissible subset-sum witness and treats the selector complement by an exact support-weighted diagonal calculation.

1.2 Logical dependencies

The argument has four independent layers: the selected twisted-moment estimate; the Euler-product and inverse-Laplace derivation of its leading diagonal; exact rational enclosures of the resulting compact integrals; and the strict Levinson–Conrey comparison. Each layer is proved below; the machine audits require all thirty signed classes and never interpret a missing class as zero.

1.3 Notation

Throughout,

$$R = \frac{13}{10}, \qquad \theta = \frac{29}{50}, \qquad \rho = \frac{200}{203}, \qquad x_* = \frac{3}{203}.$$

The actual analytic mollifier has length $T^{\theta-\delta}$ for fixed $\delta > 0$, and the endpoint $\delta \downarrow 0$ is taken only after $T \to \infty$. The retained selector is W, its complement is $S_D = 1 - W$, the formal unfiltered hybrid is A, and its deleted marked component is D. The actual mollifier is $A - D$. Every terminating decimal in the certificate denotes the corresponding exact rational number.

2 The analytic reduction and the selected long mollifier

Throughout this section,

$$\sigma_0 = \frac{1}{2} - \frac{R}{\log T}, \qquad R = \frac{13}{10},$$

and all constants implicit in O- and $\ll$-notation may depend on the fixed polynomials, smooth weights, and endpoint buffers. Every terminating decimal below denotes the corresponding exact rational number.

2.1 The Levinson–Conrey implication

We use the following standard form of the Levinson–Conrey reduction.

Theorem 2.1 (Levinson–Conrey reduction)**.** *Let $R > 0$, let $Q \in \mathbb{R}[x]$ satisfy*

$$Q(0) = 1, \qquad Q'(x) = Q'(1-x),$$

and put

$$V(s) = Q\Big(-\frac{1}{\log T}\frac{d}{ds}\Big)\,\zeta(s).$$

Let

$$\psi(s) = \sum_{n \le T^{\vartheta}} a(n) n^{\sigma_0 - 1/2 - s}, \qquad 0 < \vartheta < 1,$$

where $a(1) = 1$, the coefficients are real, and $a(n) \ll_\varepsilon n^\varepsilon$ for every $\varepsilon > 0$. If, for some fixed $c > 0$,

$$\frac{1}{T}\int_T^{2T} |V(\sigma_0 + it)\psi(\sigma_0 + it)|^2 \, dt \le c + o(1), \tag{LC.1}$$

then, with

$$\kappa = \liminf_{T\to\infty} \frac{N_0(T)}{N(T)},$$

one has

$$\kappa \ge 1 - \frac{1}{R}\log c. \tag{LC.2}$$

The same conclusion follows if the estimate is first proved with fixed smooth nonnegative height weights and then transferred to the indicator of $[T, 2T]$ by a smooth majorant.

Theorem 2.1 is Levinson's method in the form used by Conrey [2] and, with the same hypotheses on Q and on $a(n)$, by Pratt, Robles, Zaharescu, and Zeindler [8]; we use it verbatim and prove no part of it here. For the reader's orientation, the two steps are: (i) the functional equation, Littlewood's lemma, and the argument principle relate $N(T) - N_0(T)$ to the zero count and argument variation of V in a rectangle with left edge $\Re s = \sigma_0$, and bound the latter in terms of $\frac{1}{2\pi}\int_T^{2T} \log|V\psi(\sigma_0+it)|\,dt$; and (ii) the arithmetic–geometric mean inequality replaces $\frac{1}{T}\int \log|V\psi|^2$ by $\log(\frac{1}{T}\int |V\psi|^2)$, which is where (LC.1) enters. The hypothesis $\vartheta < 1$ is used only in step (i), to guarantee $\psi \ll T^{\vartheta/2+\varepsilon}$ on the relevant contour so that the horizontal edges contribute $o(N(T))$; it is not a constraint coming from the mean-value estimate, which is supplied separately by Theorem 2.2. In particular $\vartheta = 29/50$ is admissible in Theorem 2.1.

Since $R = 13/10$, proving Theorem 1.1 amounts, by (LC.2), to the following strict inequality.

$$c < \exp\left(\left(1 - \frac{421}{1000}\right)R\right) = \exp\left(\frac{7527}{10000}\right). \tag{LC.3}$$

2.2 The coefficient and the marked-subset selector

Set

$$\vartheta_* = \frac{29}{50}, \qquad \rho = \frac{200}{203}, \qquad x_* = \frac{3}{203}.$$

For the limiting calculation let $Y = T^{\vartheta_*}$. In the actual asymptotic theorem we instead use

$$\vartheta = \vartheta_* - \delta, \qquad Y = T^{\vartheta}, \qquad \delta > 0 \tag{LC.4}$$

with δ fixed until after $T \to \infty$. For

$$x_n = 1 - \frac{\log n}{\log Y}$$

and $k \in \{1, 2, 4, 5\}$, define

$$\widetilde{P}_k(x) = \begin{cases} P_k((x - x_*)/\rho), & x \ge x_*, \\ 0, & x < x_*. \end{cases} \tag{LC.5}$$

Since $\vartheta_*\rho = 4/7$, these four pieces have limiting length $T^{4/7}$; for fixed $\delta > 0$ their length is strictly smaller.

The polynomials are

$$\begin{aligned}
P_1(x) &= .2544x + 1.2742x^2 + .5800x^3 - 1.7182x^4 + .6096x^5, \\
P_2(x) &= .6834x + 7.3227x^2 - 22.2752x^3 + 30.4150x^4 - 15.2953x^5, \\
P_3(x) &= 4.2899x - 34.9701x^2 + 98.8882x^3 - 105.1306x^4 + 32.4893x^5, \\
P_4(x) &= -7.3798x + 60.4440x^2 - 123.7661x^3 + 70.1589x^4 - .7105x^5, \\
P_5(x) &= 9.2239x - 40.1893x^2 + 38.6342x^3 - .4583x^4 + .3503x^5.
\end{aligned}$$

Thus $P_1(0) = 0$, $P_1(1) = 1$, and $P_k(0) = 0$ for $k \ge 2$. Let L_j denote the Legendre polynomial of degree j, and set

$$Q(z) = 1 + .5550(L_1(1-2z) - 1) - .0498(L_3(1-2z) - 1) + .0039(L_5(1-2z) - 1). \tag{LC.6}$$

Because every Legendre degree used here is odd,

$$Q(0) = 1, \qquad Q'(z) = Q'(1-z)$$

as exact polynomial identities.

For a labelled factorization $mq_1q_2q_3 = n$, put

$$t_0 = \frac{\log m}{\log Y}, \qquad t_i = \frac{\log q_i}{\log Y} \quad (1 \le i \le 3).$$

Define

$$\mathcal{S} = \left[\frac{8}{29}, \frac{13}{29}\right] \cup \left[\frac{16}{29}, \frac{21}{29}\right] \tag{LC.7}$$

and let $W(t_1, t_2, t_3)$ be the indicator of the condition

$$\sum_{i \in I} t_i \in \mathcal{S} \quad \text{for at least one nonempty } I \subseteq \{1, 2, 3\}. \tag{LC.8}$$

The limiting coefficient is

$$\begin{aligned} a(n) &= \mu(n)\widetilde{P}_1(x_n) \\ &\quad + \sum_{k \in \{2,4,5\}} \frac{(-1)^k (\mu * \Lambda^{*k})(n)}{(\log Y)^k} \widetilde{P}_k(x_n) \\ &\quad + \frac{(-1)^3 P_3(x_n)}{(\log Y)^3} \sum_{mq_1q_2q_3 = n} \mu(m)\Lambda(q_1)\Lambda(q_2)\Lambda(q_3)W(t_1, t_2, t_3). \end{aligned} \tag{LC.9}$$

The three Λ-slots are labelled; in particular, no factor $1/3!$ occurs. All powers of $\log Y$ in (LC.9) use the master-length normalization. The factor $(-1)^k$ is the standard Feng convention and is not absorbed into P_k: differentiating k zeta-ratio markers at zero produces $(-1)^k(\mu * \Lambda^{*k})$. This convention is also the one used by the one-variable V_m compiler in Section 3. Shortening a piece changes its support and the argument of P_k, but does not replace $(\log Y)^{-k}$ by $(\log Y^\rho)^{-k}$.

For the actual coefficient fix also a radial buffer $\varepsilon_r > 0$. Choose $\chi_{\varepsilon_r} \in C_c^\infty(\mathbb{R})$ with

$$0 \le \chi_{\varepsilon_r} \le 1, \qquad \chi_{\varepsilon_r}(q) = 1 \ (0 \le q \le 1 - 2\varepsilon_r), \qquad \chi_{\varepsilon_r}(q) = 0 \ (q \ge 1 - \varepsilon_r), \qquad \|\chi'_{\varepsilon_r}\|_\infty \le C/\varepsilon_r.$$

In the selected summand replace W by a fixed smooth finite tensor-product approximation, in C^1 on the relevant compact set, to

$$\chi_{\varepsilon_r}(t_0 + t_1 + t_2 + t_3) W_\eta(t_1, t_2, t_3).$$

Denote the chosen approximation by $W_{\eta, \varepsilon_r, \mathcal{P}}$. Each tensor box is relative to the nonnegative orthant; its relative closure lies in the tuple simplex, is a positive distance from the slanted face $t_0 + t_1 + t_2 + t_3 = 1$, and lies inside one fixed buffered witness slab from (LC.8). The parameters $\eta, \varepsilon_r, \delta$, and the finite tensor mesh $\mathcal{P}$ are fixed before $T \to \infty$. Finally,

$$\psi(s) = \sum_{n \le Y} a(n) n^{\sigma_0 - 1/2 - s}. \tag{LC.10}$$

The coefficient satisfies $a(1) = 1$ and $a(n) \ll_\varepsilon n^\varepsilon$.

2.3 A selected twisted-moment theorem

We isolate the analytic input that permits the long selected piece. The diagonal expression below is the one in Theorem 4.1 of [8]; it is understood by analytic continuation when $\alpha + \beta = 0$.

Theorem 2.2 (Selected twisted second moment)**.** *Fix $\delta > 0$ and a positive selector buffer. Let $\vartheta = 29/50 - \delta$ and $Y = T^{\vartheta}$. Let $a_{\mathrm{sel}}(n)$ be a finite linear combination of smooth product-box pieces of*

$$\frac{(-1)^3 P(1-\log n/\log Y)}{(\log Y)^3} \sum_{mq_1q_2q_3=n} \mu(m)\Lambda(q_1)\Lambda(q_2)\Lambda(q_3)\mathcal{W}(t_0,t_1,t_2,t_3), \tag{SM.1}$$

where $a_{\mathrm{sel}}(n) = 0$ for $n > Y$, and the relative closure of every product box is contained in the tuple simplex

$$t_i \geq 0, \qquad t_0+t_1+t_2+t_3 \leq 1,$$

has positive distance from its slanted face $t_0+t_1+t_2+t_3 = 1$, and is contained in a single buffered witness slab from (LC.8): that is, one fixed nonempty $I \subseteq \{1,2,3\}$, attached to the box, satisfies $\sum_{i\in I} t_i$ in one fixed buffered component of $\mathcal{S}$ throughout the closure. Let $b(n) \ll_\varepsilon n^\varepsilon$ be supported on $n \leq T^{v_0}$ with $v_0 \leq \vartheta$. Put

$$\mathcal{A}(s) = \sum_n a_{\mathrm{sel}}(n)n^{-s}, \qquad \mathcal{B}(s) = \sum_n b(n)n^{-s}.$$

If Φ is a fixed smooth function of compact support in $(0,\infty)$, then, uniformly for $|\alpha|, |\beta| \leq C/\log T$,

$$\begin{aligned}\int_{-\infty}^{\infty} \zeta\Big(\frac{1}{2}+\alpha+it\Big)\zeta\Big(\frac{1}{2}+\beta-it\Big)\mathcal{A}\Big(\frac{1}{2}+it\Big)\overline{\mathcal{B}\Big(\frac{1}{2}+it\Big)}\Phi\Big(\frac{t}{T}\Big)\,dt&\\ = \mathcal{M}_{\alpha,\beta}(a_{\mathrm{sel}},b;\Phi) + O(T^{1-\epsilon_0}),&\end{aligned} \tag{SM.2}$$

for some $\epsilon_0 > 0$, where

$$\begin{aligned}\mathcal{Z}_{\alpha,\beta}(d,e;t) := \zeta(1+\alpha+\beta)&\\ + \zeta(1-\alpha-\beta)\left(\frac{2\pi de}{t(d,e)^2}\right)^{\alpha+\beta}&,\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}(a,b;\Phi) = \sum_{d,e}\frac{a(d)\overline{b(e)}}{[d,e]}\frac{(d,e)^{\alpha+\beta}}{d^\alpha e^\beta}&\\ \times\int_{-\infty}^{\infty}\mathcal{Z}_{\alpha,\beta}(d,e;t)\Phi\Big(\frac{t}{T}\Big)\,dt.&\end{aligned} \tag{SM.3}$$

The conclusion remains valid if the selected coefficient is on the second side. It therefore applies both to the selected piece against itself and to its cross with every piece of length at most $T^{4/7-\epsilon}$.

Proof. Here and below a product box is relative to the nonnegative orthant:

$$B = \mathbb{R}^4_{\geq 0} \cap (J_0 \times J_1 \times J_2 \times J_3),$$

with closure also taken relative to that orthant. Thus a box may meet a coordinate face (as it must when an atom equals 1); the hypothesis only requires positive distance from the slanted face $t_0+t_1+t_2+t_3 = 1$ and from the endpoints of its fixed witness slab.

The proof follows the off-diagonal argument of Theorem 4.1 of [8], replacing its Heath–Brown partition by a partition already present in the marked tuple. We give all exponent checks because this is the only point at which the length exceeds 4/7.

The bilinear input and its parameters. We use [5, Lemma 1] exactly in the form in which it is applied inside the proof of [8, Theorem 4.1], and we record that form here so that the exponent bookkeeping below is self-contained. That lemma bounds an average of Kloosterman-type sums attached to four *length parameters*, each required only to be at least 1, together with a fixed positive integer multiplier; PRZZ instantiate the four length parameters as

$$E, \qquad F, \qquad A_{\mathrm{freq}}, \qquad V,$$

and the multiplier as h. Here E and F are the lengths of the two convolution factors into which the *left* (mollifier) coefficient is split, A_{freq} is the length of the shifted-frequency range produced by the PRZZ decomposition, and V is the length of the right-hand Dirichlet polynomial. We write throughout

$$U = EF,$$

so that U is the total length of the left coefficient; U is not an independent parameter, and E designates the *shorter* of the two left factors. With this dictionary the bound of [5, Lemma 1] is the sum of the seven monomials (SM.4) below. Nothing in the present argument changes the lemma, its hypotheses, or the way PRZZ apply it; the only new point is *which* factorization $U = EF$ is used, and that is the content of (SM.7).

Fix one product box and write

$$q = t_0 + t_1 + t_2 + t_3 \le 1, \qquad u_\delta = \vartheta q, \qquad v_\delta \le \vartheta.$$

If $q \le 21/29$, then, for the actual fixed $\vartheta = 29/50 - \delta$,

$$u_\delta + v_\delta \le \vartheta\left(1 + \frac{21}{29}\right) = 1 - \frac{50}{29}\delta < 1.$$

After choosing the harmless bookkeeping exponent smaller than this fixed gap, the nonzero shifted-frequency range in the PRZZ decomposition is empty. For the remaining dyadic blocks, where (q>21/29), using $u_0 = \vartheta_* q$ and $v_0 = \vartheta_*$ only enlarges every relevant monomial.

Suppose henceforth that $q > 21/29$. Split the tuple into two fixed convolution factors, orient them so that the shorter factor has length $E = T^e$ and the longer has length $F = T^{u-e}$, and write $e = \vartheta_* z$. Before common-divisor extraction, put $H = 0$, $U = EF = T^u$, and $V = T^v$. With the dictionary fixed above, the seven terms supplied by [5, Lemma 1] are

$$\begin{aligned}
&E^{-1/2}A_{\mathrm{freq}}^{1/2}UV, && h^{1/4}A_{\mathrm{freq}}UV^{1/2},\\
&h^{1/4}E^{1/4}A_{\mathrm{freq}}^{3/4}UV^{1/2}, && h^{1/4}E^{-1/4}A_{\mathrm{freq}}U^{3/4}V,\\
&h^{1/2}E^{1/4}A_{\mathrm{freq}}U^{3/4}V^{3/4}, && h^{1/4}E^{1/4}A_{\mathrm{freq}}^{1/2}U^{3/4}V,\\
&h^{1/2}E^{3/4}A_{\mathrm{freq}}^{1/2}U^{3/4}V^{3/4}. &&
\end{aligned} \tag{SM.4}$$

Here (SM.4) is a seven-term majorant, not a literal term-for-term expansion of the primary DI display. Before domination that display is

$$(EFA_{\mathrm{freq}}V)^{1/2}\left\{(FV)^{1/2} + (E + A_{\mathrm{freq}})^{1/4}\left[FV(A_{\mathrm{freq}} + hE)(V + hE^2) + hE^2F^2A_{\mathrm{freq}}\right]^{1/4}\right\}.$$

Expanding the positive sums gives ten monomials. Four are exactly terms 2, 3, 6, and 7 of (SM.4); the term $E^{-1/4}A_{\mathrm{freq}}U^{3/4}V$ is bounded by term 4 because $h \ge 1$. Of the remaining five, the unweighted $A_{\mathrm{freq}}^{3/4}U^{3/4}V$ is bounded by its $h^{1/4}$-weighted companion, and $h^{1/4}E^{1/4}A_{\mathrm{freq}}U^{3/4}V^{3/4}$ is bounded by term 5. If $A_{\text{freq}} \geq E$, the other three are bounded by terms 4 and 5; if $A_{\text{freq}} \leq E$, they are bounded by terms 6 and 7. This is the precise equality/dominance count behind the seven-term display. Here

$$A_{\text{freq}} = T^{H+u+v-1+o(1)}.$$

The outer PRZZ integral contributes $1/A_{\text{freq}}$. Consequently the seven exponents after that division are

$$\begin{aligned}
\lambda_1 &= \frac{1+u+v-H-e}{2},\\
\lambda_2 &= \frac{H}{4} + u + \frac{v}{2},\\
\lambda_3 &= \frac{1}{4} + \frac{3u}{4} + \frac{v}{4} + \frac{e}{4},\\
\lambda_4 &= \frac{H}{4} - \frac{e}{4} + \frac{3u}{4} + v,\\
\lambda_5 &= \frac{H}{2} + \frac{e}{4} + \frac{3u}{4} + \frac{3v}{4},\\
\lambda_6 &= \frac{1}{2} + \frac{u}{4} + \frac{v}{2} + \frac{e}{4} - \frac{H}{4},\\
\lambda_7 &= \frac{1}{2} + \frac{u}{4} + \frac{v}{4} + \frac{3e}{4}.
\end{aligned} \tag{SM.5}$$

At $H = 0$, $u = (29/50)q$, $v = 29/50$, and $e = (29/50)z$, the seven conditions $\lambda_j \leq 1$ are

j	condition
1	$z \geq q - 21/29,$
2	$q \leq 71/58,$
3	$z \leq 121/29 - 3q,$
4	$z \geq 3q - 84/29,$
5	$z \leq 113/29 - 3q,$
6	$z \leq 42/29 - q,$
7	$z \leq (71/29 - q)/3.$

(SM.6)

They all follow from

$$q - \frac{21}{29} \leq z \leq \min\left(\frac{q}{2}, \frac{42}{29} - q\right). \tag{SM.7}$$

Indeed, rows 2, 3, and 5 are immediate on $21/29 \leq q \leq 1$, and row 1 implies row 4 because

$$q - \frac{21}{29} \geq 3q - \frac{84}{29} \qquad (q \leq 1).$$

For row 7, the bound $z \leq q/2$ suffices for $q \leq 142/145$, while row 6 suffices for $q \geq 55/58$; these two ranges overlap.

We now prove that every selected box supplies a fixed factorization obeying (SM.7). Let

$$s = \sum_{i \in I} t_i$$

be a selected subset mass. If $s \in [8/29, 13/29]$ and $s \leq q/2$, use s as the shorter mass. Its lower bound follows from $q \leq 1$, and its upper bound follows from $s \leq q/2$ when $q \leq 28/29$ and from

$$s \leq \frac{13}{29} \leq \frac{42}{29} - q$$

when $q \geq 28/29$. If instead $s > q/2$, use the complementary mass $q - s$. Then $q < 26/29$, so the active upper bound is $q/2$, and

$$q - s \geq q - \frac{13}{29} \geq q - \frac{21}{29}, \qquad q - s < \frac{q}{2}.$$

If $s \in [16/29, 21/29]$, then $s > q/2$ because $q \leq 1$, so again use $q - s$. The lower bound is

$$q - s \geq q - \frac{21}{29}.$$

For $q \leq 28/29$ the upper bound $q - s \leq q/2$ is automatic. For $q \geq 28/29$, the remaining upper bound is equivalent to

$$s \geq 2q - \frac{42}{29};$$

the right side is largest at $q = 1$, where it equals $16/29$. This proves (SM.7) in every case. A positive selector buffer and $\vartheta < \vartheta_*$ make every required inequality strict.

There is one orientation wall to record explicitly. If $s = q/2$, the second component of the selector is impossible, since it would give $q = 2s \geq 32/29 > 1$. Hence $s \in [8/29, 13/29]$ and $q = 2s \leq 26/29$. At this wall the two arithmetic convolution factors have equal length, and direct substitution of $z = s = q - s = q/2$ in (SM.6) makes all seven inequalities strict; for the two potentially active facets the normalized margins are at least $8/29$ in row 1 and $3/29$ in row 6. By continuity, a sufficiently small neighborhood of the compact buffered wall permits either fixed factor to be used. We therefore refine the product-box cover into boxes lying on one side of $s = q/2$ and boxes in these wall neighborhoods. On the latter fix either arithmetic split; the raw DI lemma does not require an a priori length ordering. After the usual dyadic decomposition the two factors may be interchanged when designating the shorter length as E. Thus "orient" above is a fixed boxwise/dyadic designation, not a pointwise existential choice inside the DI sum.

We next justify that the chosen factorization remains an exact convolution after PRZZ extracts common divisors. Let

$$f_i(r) = (\log r)^{j_i} \Lambda(r) \omega_i(r) \quad (1 \leq i \leq 3),$$

where the ω_i are the fixed smooth one-dimensional box weights, and let ω_0 be the weight on the Möbius atom. For coprime positive integers A and B, define

$$C_{\boldsymbol{\omega}}(AB) = \sum_{mq_1q_2q_3 = AB} \mu(m)\omega_0(m) \prod_{i=1}^{3} f_i(q_i).$$

Every nonzero $f_i(q_i)$ is supported on a prime power. Hence there is a unique allocation

$$J = \{i : q_i \mid A\},$$

and the remaining marked atoms divide B. Also $m = m_A m_B$ uniquely, with $m_A \mid A^\infty$, $m_B \mid B^\infty$, and $\mu(m) = \mu(m_A)\mu(m_B)$, because $(A, B) = 1$. It is important not to Mellin-invert $\omega_0(m_A m_B)$ at this point: although that identity would be exact after integration, an individual Mellin term would no longer display the compact support of the residual Möbius variable which is needed in the bilinear estimate. Instead, for a fixed A-side factor m_A, define the supported arithmetic function

$$g_{m_A}(r) = \mu(r)\omega_0(m_A r).$$

Conditioning on the complete factorization of A gives the exact identity

$$\boxed{C_{\boldsymbol{\omega}}(AB) = \sum_{J\subseteq\{1,2,3\}} \sum_{m_A \prod_{i\in J} q_i = A} \mu(m_A) \prod_{i\in J} f_i(q_i) \left(g_{m_A} * \mathop{*}_{i\notin J} f_i \right)(B).} \tag{SM.8}$$

An empty marked convolution means the delta mass at 1. Formula (SM.8) is exact and includes prime powers; no squarefree approximation has been made. Indeed, every tuple on the left has exactly one allocation J; after $m = m_A m_B$, its A-variables occur in the inner coefficient of (SM.8), while the sum over m_B and the remaining marked variables is precisely the displayed convolution at B. Conversely every term on the right reconstructs one tuple on the left.

In the PRZZ extraction one first removes the common divisor d and then writes the old left variable as hn, where

$$h \mid d^\infty, \qquad (n, dh) = 1.$$

Thus (SM.8) applies with $A = dh$ and $B = n$. A marked atom may straddle d and h separately—for example $q_i = p^2$, $d = p$, $h = p$—but it cannot straddle the coprime pair dh, n. If I is the witness selected before extraction, the residual n-coefficient factors exactly as

$$\left(\mathop{*}_{i\in I\setminus J} f_i \right) * \left(g_{m_A} * \mathop{*}_{i\in I^c\setminus J} f_i \right), \qquad I^c = \{1,2,3\} \setminus I. \tag{SM.9}$$

The two factors are divisor-bounded and may be interchanged. More importantly, their supports are explicit. If the Möbius box weight is supported where $\log m/\log Y \in K_0$, then, for fixed m_A, g_{m_A} is supported where

$$\frac{\log r}{\log Y} \in K_0 - \frac{\log m_A}{\log Y}, \qquad r \geq 1.$$

This is a translate of the original compact interval, with the same smooth norms in the logarithmic coordinate. Explicitly, if $\omega_0(r) = \Omega_0(\log r/\log Y)$, then for every fixed k,

$$\left(r\frac{d}{dr} \right)^k \omega_0(m_A r) = \frac{1}{(\log Y)^k} \Omega_0^{(k)} \left(\frac{\log m_A + \log r}{\log Y} \right),$$

uniformly in m_A. The marked factors retain their original box intervals. Thus an ordinary dyadic subdivision of (SM.9) has two fixed supported factors to which the raw Deshouillers–Iwaniec lemma applies term by term.

For each J, the outside sum over $m_A \prod_{i\in J} q_i = A$ has at most a fixed divisor-function number of terms, and its logarithmic weights give

$$\sum_{m_A \prod_{i\in J} q_i = A} |\mu(m_A)| \prod_{i\in J} |f_i(q_i)| \ll \tau(A)^C (\log T)^C.$$

This is absorbed by the factor $\tau(dh)^C T^\varepsilon$ already retained in (SM.13). The outer radial cutoff and polynomial weight are then separated by the usual Mellin inversion and binomial theorem, after the supported factors in (SM.9) have been fixed. Thus no pointwise existential selector and no support-destroying Mellin term occurs inside the Deshouillers–Iwaniec sum.

It remains to verify the exponent inequalities after this extraction. Write

$$d = T^D, \qquad h = T^H,$$

up to harmless dyadic factors. If (u_0, v_0, e_0) are the original exponents and e is the new shorter-factor exponent, then

$$u = u_0 - D - H, \qquad v = v_0 - D, \qquad 0 \le g := e_0 - e \le D + H. \tag{SM.10}$$

To see the last inequality with the support information now explicit, let x, y be the original two factor exponents and let $r_1, r_2 \ge 0$ be the exponents of the variables allocated to dh from the two factors. Then $r_1 \le x$, $r_2 \le y$, $r_1 + r_2 = D + H$, $e_0 = \min(x, y)$, and $e = \min(x - r_1, y - r_2)$. Hence

$$e_0 - (D + H) \le e \le e_0,$$

which is exactly (SM.10). Subtracting the original values from (SM.5) gives

$$\begin{array}{c|c} j & \lambda_j - \lambda_j^0 \\ \hline 1 & (g - 2D - 2H)/2, \\ 2 & -3D/2 - 3H/4, \\ 3 & -D - 3H/4 - g/4, \\ 4 & g/4 - 7D/4 - H/2, \\ 5 & -g/4 - 3D/2 - H/4, \\ 6 & -g/4 - 3D/4 - H/2, \\ 7 & -3g/4 - D/2 - H/4. \end{array} \tag{SM.11}$$

Using $g \le D + H$, the seven terms gain at least

$$\begin{array}{c|ccccccc} j & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline d^{-a_j} & d^{-1/2} & d^{-3/2} & d^{-1} & d^{-3/2} & d^{-3/2} & d^{-3/4} & d^{-1/2} \\ h^{-b_j} & h^{-1/2} & h^{-3/4} & h^{-3/4} & h^{-1/4} & h^{-1/4} & h^{-1/2} & h^{-1/4}. \end{array} \tag{SM.12}$$

The original PRZZ factor $1/d$ lies outside this table. If the buffered pre-extraction exponent is at most $1 - \gamma$, the j-th contribution is therefore bounded by

$$\frac{T^{1-\gamma+\varepsilon}}{d} \tau(dh)^C d^{-a_j} h^{-b_j}. \tag{SM.13}$$

For every fixed $\beta > 0$,

$$\begin{aligned} \sum_{h \mid d^\infty} \frac{\tau(dh)^C}{h^\beta} &= \prod_{p^r \| d} \sum_{k \ge 0} (r + k + 1)^C p^{-\beta k} \\ &\ll_{C,\beta,\varepsilon} d^\varepsilon. \end{aligned}$$

Since every $a_j > 0$ and every $b_j \ge 1/4$, after absorbing the small shift powers one obtains

$$\sum_d d^{-1-a_j+\varepsilon} \sum_{h \mid d^\infty} \tau(dh)^C h^{-b_j+\varepsilon} \ll 1. \tag{SM.14}$$

Thus the complete extracted contribution is $O(T^{1-\gamma/2})$ after choosing the bookkeeping ε sufficiently small.

Lemma 1 of [5] requires its four length parameters only to be at least 1. Under the PRZZ identification of those parameters with E, F, A_{freq}, V, and of its fixed multiplier with h, the same bound therefore applies when $E = 1$ or $F = 1$. In that case one residual factor in (SM.9) is simply the delta mass at 1. If $A_{\mathrm{freq}} < 1$, the shifted-frequency range is empty; the endpoint $A_{\mathrm{freq}} = 1$ is permitted by the lemma.

The other preliminary errors in the proof of Theorem 4.1 of [8] use only divisor bounds. In the notation above, the bounds inherited from the Pratt–Robles reduction are

$$O(T^{1/2+\varepsilon}), \qquad O(Y^{1/2}T^{1/2+\varepsilon}), \qquad O(1 + Y^2 T^{-3/2+\varepsilon}), \qquad O(YT^{\varepsilon}),$$

for, respectively, the diagonal gamma-ratio replacement, the auxiliary gamma-kernel replacement, the phase/kernel Taylor remainder, and the smooth-kernel remainder. The large determinant-shift tail before Poisson summation and the large nonzero Poisson-frequency tail afterward are each $O_A(T^{-A})$ after sufficiently many integrations by parts; the zero-mode completion costs $O(T^{\varepsilon})$. All these estimates remain valid for unequal sides by using their maximum length. Since $Y = T^{29/50-\delta}$, every displayed term is $o(T)$ after choosing the bookkeeping ε below the fixed δ-gap. Hence the seven terms above are the only new off-diagonal terms requiring a length calculation.

Every expression in (SM.5) is nondecreasing in the other initial length v_0. The calculation with $v_0 = 29/50$ therefore dominates all unequal crosses, including every 29/50-by-4/7 cross. If the selected coefficient occurs on the second side, conjugate the moment and exchange the two Dirichlet polynomials.

Finally, the estimate is uniform in a fixed $C/\log T$ polydisc in (α, β). By Cauchy's formula, applying the fixed polynomial Q costs only $(\log T)^{O(1)}$, which the power saving absorbs. This proves (SM.2). $\square$

2.4 Sharp selector, endpoint, and height limits

For $\eta > 0$, replace $\mathcal{S}$ by

$$\mathcal{S}_\eta = \left[\frac{8}{29} + 2\eta, \frac{13}{29} - 2\eta\right] \cup \left[\frac{16}{29} + 2\eta, \frac{21}{29} - 2\eta\right].$$

Choose a smooth function $W_\eta(t_1, t_2, t_3)$, independent of t_0, with $0 \le W_\eta \le 1$, supported where some subset sum lies in $\mathcal{S}_\eta$, and equal to 1 where some subset sum lies in the corresponding union with 3η in place of 2η. Compactness gives a finite cover only after imposing the radial buffer; this is why ε_r was retained above. More explicitly, on the nonnegative orthant the support of

$$\chi_{\varepsilon_r}(t_0 + t_1 + t_2 + t_3) W_\eta(t_1, t_2, t_3)$$

is compact and has positive distance from the slanted face $t_0 + t_1 + t_2 + t_3 = 1$. At each point of this support choose one subset witness and one component of $\mathcal{S}_\eta$. By the selector buffer and continuity, a sufficiently small product neighborhood has its closure inside both the tuple simplex and that same witness slab. These product neighborhoods and their closures are relative to the nonnegative orthant, so coordinate faces remain allowed. Refine them also by the two orientation regions and the compact wall neighborhoods described in the proof of Theorem 2.2. These refined sets are open; choose smaller product neighborhoods whose closures lie in one refined set. Compactness gives a finite subcover. A subordinate smooth partition and tensor approximation on each of these boxes give exactly the class of coefficients required by Theorem 2.2; in particular the subset I used in (SM.9) is fixed on each summand.

The leading diagonal supplied by (SM.3) is a finite sum of integrals of the form

$$\int_\Omega p(\lambda) e^{\ell(\lambda)} W_{\eta,L}(\boldsymbol{x}) W_{\eta,R}(\boldsymbol{y})\, d\lambda, \tag{SM.15}$$

or the corresponding one-sided cross integral. Here Ω is a compact flow polytope, p is a fixed polynomial, and ℓ is affine. The selector depends only on the three marked coordinates. The sole spatial operator in the support-weighted polar kernel is $a - \partial_{x_0}$, so it never differentiates a selector face. Moreover, $P_3(0) = 0$ removes the radial endpoint distribution. Thus (SM.15) is an ordinary finite Lebesgue integral.

The selector boundary is a finite union of affine hyperplanes. Under any labelled flow map, every nonempty marked subset sum pulls back to a nonconstant 0–1 linear form in the marked flow variables: a matched coordinate is one of the z_{ij}, and an unmatched coordinate is one of the u_i or v_j. Its equality to any of $8/29, 13/29, 16/29, 21/29$ is therefore null for the ambient Lebesgue measure in that flow. Any lower-dimensional intersection introduced by a cap or selector-cell clipping is already null for this same ambient measure; no relative Hausdorff measure is used. Hence $W_\eta \to W$ almost everywhere as $\eta \downarrow 0$. Since the domains are compact and $0 \le W_\eta \le 1$, dominated convergence gives the sharp diagonal. The same finite integral is continuous as $\delta \downarrow 0$, as the shortened cutoff tends to its endpoint, and as the tensor approximation is refined. It is also continuous as $\varepsilon_r \downarrow 0$. Indeed, with

$$p(q) = P_3(1-q)\mathbf{1}_{\{0 \le q \le 1\}}, \qquad p_{\varepsilon_r}(q) = p(q)\chi_{\varepsilon_r}(q),$$

we have $\|p_{\varepsilon_r} - p\|_{W^{1,1}([0,\infty))} \to 0$: the difference is supported in an interval of length $2\varepsilon_r$. On that interval $P_3(0) = 0$ and the mean-value theorem give $|p(q)| \ll \varepsilon_r$; hence

$$\int |p(q)\chi'_{\varepsilon_r}(q)|\, dq \ll \varepsilon_r \cdot \varepsilon_r^{-1} \cdot \varepsilon_r \ll \varepsilon_r,$$

while the remaining function and derivative terms are $O(\varepsilon_r)$ by absolute continuity. The flow formula contains only p and its first radial derivative, so this $W^{1,1}$-convergence gives the asserted diagonal continuity.

Consequently, if the certified sharp limiting constant c_* satisfies a strict inequality $c_* < c_{\text{target}}$, first choose fixed $\eta > 0$, $\varepsilon_r > 0$, $\delta > 0$, and a fixed finite tensor mesh so that the resulting diagonal constant is still below c_{target}. Only then take $T \to \infty$. Theorem 2.2 supplies a fixed power saving for this one fixed mollifier. No estimate uniform in a joint limit $T \to \infty$, $\eta, \varepsilon_r, \delta \downarrow 0$ is used.

The twisted-moment theorem is stated with a smooth height weight. Its proof is unchanged after a fixed rescaling, or a finite smooth partition, for any fixed compact support in $(0,\infty)$. Choose a nonnegative smooth majorant of $\mathbf{1}_{[1,2]}$, supported in $[1-\epsilon, 2+\epsilon]$, with total mass $1+O(\epsilon)$. The strict margin in the final numerical inequality permits one sufficiently small positive ϵ to be chosen once and for all. Freeze that majorant, including all of its derivative norms, before applying the $T \to \infty$ asymptotic. Its fixed mass factor is then absorbed by the strict margin and yields the unweighted estimate required in Theorem 2.1. A standard dyadic summation gives the corresponding global form of the Levinson–Conrey inequality.

3 The support-weighted diagonal from Euler products

In this section $\theta := \vartheta_* = 29/50$ denotes the limiting mollifier exponent and

$$L = \log Y = \theta \log T.$$

The actual mollifier uses $\theta - \delta$ until after $T \to \infty$, as in (LC.4); the formulas here are the subsequent $\delta \downarrow 0$ limits. If S is a bounded symmetric function of the three marked coordinates, define the support-weighted P_3-coefficient by

$$C_S(n) = \frac{(-1)^3 P_3(1 - \log n/L)}{L^3} \sum_{mq_1q_2q_3=n} \mu(m)\Lambda(q_1)\Lambda(q_2)\Lambda(q_3) S\left(\frac{\log q_1}{L}, \frac{\log q_2}{L}, \frac{\log q_3}{L}\right) \qquad (1)$$

for $n \le Y$, and put $C_S(n) = 0$ for $n > Y$. The variables q_1, q_2, q_3 are labelled, as they are in the three auxiliary Λ-derivatives below. Thus there is no factor $1/3!$ in (1).

Let W be the retained marked-coordinate selector and write $S_D = 1 - W$. If A_{sh} denotes the sum of the four shortened pieces, set

$$A = A_{\mathrm{sh}} + C_1, \qquad D = C_{S_D}, \qquad A - D = A_{\mathrm{sh}} + C_W. \tag{2}$$

Here A is an unfiltered coefficient used only for diagonal bookkeeping. It is not asserted to possess a satisfactory off-diagonal estimate at length Y.

3.1 The normalization of the diagonal form

The bilinear form $\mathcal{D}_Q$ used throughout the paper is fixed by the following definition. All later numerical statements are statements about this normalization and no other.

Definition 3.1 (Diagonal form and its normalization)**.** Let Φ be a fixed nonnegative, nonzero smooth height weight of compact support in $(0, \infty)$, as permitted in Theorem 2.2, and put $\|\Phi\|_1 = \int_{\mathbb{R}} \Phi(x)\,dx$. For arithmetic coefficients f, g supported on $n \le Y$ let $\mathcal{M}_{\alpha,\beta}(f, g; \Phi)$ be the main term (SM.3), and set

$$\mathrm{Diag}_T(f, g) = Q(-\theta\partial_a)Q(-\theta\partial_b)\mathcal{M}_{\alpha,\beta}(f, g; \Phi)\Big|_{\substack{\alpha=a/L,\ \beta=b/L, \\ a=b=-\theta R}} \tag{3}$$

the two Q-operators being applied by Cauchy's formula on fixed circles of radius $c/\log T$, as in Theorem 2.1. Define

$$\mathcal{D}_Q(f, g) = \lim_{T\to\infty} \frac{\mathrm{Diag}_T(f, g)}{T\|\Phi\|_1} \tag{4}$$

whenever the limit exists.

Three remarks fix the bookkeeping. First, the substitution $\alpha = a/L$ with $L = \theta \log T$ turns the operator $Q(-(\log T)^{-1}\partial_\alpha)$ of Theorem 2.1 into $Q(-\theta\partial_a)$, and the Levinson shift $\sigma_0 - \frac{1}{2} = -R/\log T$ into $a = -\theta R$; this is the evaluation point in (23). Second, the limit (4) exists and all powers of L cancel: the $q + r$ marker derivatives supply L^{q+r} against the normalizations L^{-q} and L^{-r} carried by the two coefficients, and the remaining power is absorbed by the simple pole of $\zeta(1 + \alpha + \beta)$. This is precisely the content of Proposition 3.9 and Corollary 3.11, and the resulting finite value is computed by (23). Third, the two summands of $\mathcal{Z}_{\alpha,\beta}$ in (SM.2) produce the two branches of (22): $\zeta(1+\alpha+\beta) = L/(a+b)+O(1)$ gives the first branch, while $\zeta(1-\alpha-\beta) = -L/(a+b)+O(1)$ together with $(2\pi de/(t(d, e)^2))^{\alpha+\beta}$ at $t \asymp T$ gives the second, carrying the factor $T^{-(\alpha+\beta)} = e^{-(a+b)/\theta}$ and the opposite sign. For the fixed smooth majorant used in the unweighted estimate, the admissible constant c of (LC.1) is $\|\Phi\|_1\mathcal{D}_Q(\psi, \psi)$. It tends to $\mathcal{D}_Q(\psi, \psi)$ when the height buffer is removed; see Proposition 3.15 below.

Proposition 3.2 (Exact diagonal bookkeeping)**.** *With* Diag_T *as in* (3)*, before taking any limit,*

$$\begin{aligned} \mathrm{Diag}_T(A - D, A - D) = {} & \mathrm{Diag}_T(A, A) - \mathrm{Diag}_T(A, D) \\ & - \mathrm{Diag}_T(D, A) + \mathrm{Diag}_T(D, D). \end{aligned} \tag{5}$$

Consequently, with $\mathcal{D}_Q$ *as in Definition 3.1,*

$$c_{\mathrm{sel}} = c_0 - 2AD + DD, \quad c_0 = \mathcal{D}_Q(A, A), \quad AD = \mathrm{Re}\,\mathcal{D}_Q(A, D), \quad DD = \mathcal{D}_Q(D, D). \tag{6}$$

For the present real coefficients the mixed pairing is real, so the Re *may be omitted.*

Proof. The operation which retains equal arithmetic indices is sesquilinear. Expanding $(A-D)\overline{(A-D)}$ therefore gives (5) identically at finite height. Hermitian symmetry gives $\mathcal{D}_Q(D,A) = \overline{\mathcal{D}_Q(A,D)}$, and division by the normalizing factor followed by passage to the leading diagonal gives (6). In particular, an off-diagonal theorem is needed only for the actual coefficient $A-D$, not for the formal term A occurring on the right of (5). □

3.2 The scaled polar coefficient and its partial matchings

To state the polar coefficient, let α, β be the two spectral shifts, let ξ, ψ be the outer Perron variables, let $\boldsymbol{\sigma}, \boldsymbol{\tau}$ be the four atom shifts on the two sides, and let $\boldsymbol{\eta}, \boldsymbol{\nu}$ be the six marker variables. The exact Euler-product justification of the polar replacement is given in the next subsection.

Scale all small variables by L:

$$\begin{aligned} a = \alpha L, \quad b = \beta L, \quad X = \xi L, \quad Z = \psi L, \\ s_i = \sigma_i L, \quad t_j = \tau_j L, \quad e_i = \eta_i L, \quad f_j = \nu_j L. \end{aligned} \tag{7}$$

Put, for $0 \le i, j \le 3$,

$$\ell_i = a + X + s_i, \qquad r_j = b + Z + t_j, \qquad d_{ij} = X + Z + s_i + t_j. \tag{8}$$

Replacing each polar zeta function by its leading pole turns the first branch, including its factor $\zeta(1+\alpha+\beta)$, into

$$\frac{\ell_0 r_0}{(a+b)d_{00}} G(\boldsymbol{e}, \boldsymbol{f}), \tag{9}$$

where

$$\begin{aligned} G = \prod_{i=1}^{3} \frac{\ell_i}{\ell_i + e_i} \prod_{j=1}^{3} \frac{r_j}{r_j + f_j} \prod_{i=1}^{3} \frac{d_{i0} + e_i}{d_{i0}} \prod_{j=1}^{3} \frac{d_{0j} + f_j}{d_{0j}} \\ \times \prod_{i,j=1}^{3} \frac{(d_{ij} + e_i)(d_{ij} + f_j)}{d_{ij}(d_{ij} + e_i + f_j)}. \end{aligned} \tag{10}$$

Proposition 3.3 (Partial-matchings formula)**.** *Let $\mathcal{M}_{3,3}$ be the set of partial matchings in the complete bipartite graph with left and right vertex sets $\{1,2,3\}$. For $M \in \mathcal{M}_{3,3}$, let $V_L(M)$ and $V_R(M)$ denote the sets of left and right vertices incident to an edge of M. Then*

$$\begin{aligned} [e_1 e_2 e_3 f_1 f_2 f_3] G = \sum_{M \in \mathcal{M}_{3,3}} \prod_{(i,j) \in M} \frac{1}{d_{ij}^2} \\ \times \prod_{i \notin V_L(M)} \left(\frac{1}{d_{i0}} - \frac{1}{\ell_i} \right) \prod_{j \notin V_R(M)} \left(\frac{1}{d_{0j}} - \frac{1}{r_j} \right). \end{aligned} \tag{11}$$

There are exactly

$$\sum_{m=0}^{3} \binom{3}{m}^2 m! = 1 + 9 + 18 + 6 = 34 \tag{12}$$

summands in (11).

Proof. For an unmatched left marker, the coefficient of e_i in its two one-sided factors is

$$[e_i] \frac{\ell_i}{\ell_i + e_i} \frac{d_{i0} + e_i}{d_{i0}} = \frac{1}{d_{i0}} - \frac{1}{\ell_i}, \tag{13}$$

and the right analogue is $d_{0j}^{-1} - r_j^{-1}$. A left–right factor satisfies the exact identity

$$\frac{(d+e)(d+f)}{d(d+e+f)} = 1 + \frac{ef}{d(d+e+f)} = 1 + \frac{ef}{d^2} + O(e^2 f + ef^2). \tag{14}$$

Only the squarefree monomial in the six marker variables is being extracted. Thus a cross factor can supply $e_i f_j / d_{ij}^2$, but no marker can be supplied twice. The selected cross factors are therefore the edges of a partial matching. Every vertex not in the matching is supplied by (13) or its right analogue. This proves (11). A matching of size m is formed by choosing m left vertices, m right vertices, and a bijection between them, giving $\binom{3}{m}^2 m!$ choices and hence (12). ☐

3.3 Real inverse Laplace transform and the reflected branch

For a bounded marked-coordinate weight S, extend

$$F_S(\boldsymbol{x}) = P_3(1 - |\boldsymbol{x}|) S(x_1, x_2, x_3) \mathbf{1}_{\{x_i \geq 0,\ |\boldsymbol{x}| \leq 1\}}, \qquad |\boldsymbol{x}| = x_0 + x_1 + x_2 + x_3, \tag{15}$$

by zero outside the simplex. Define

$$\mathcal{L}_a F = (a - \partial_{x_0}) F, \qquad \mathcal{L}_b G = (b - \partial_{y_0}) G. \tag{16}$$

The sign follows from the inverse-Laplace convention $F(x) = \int \widehat{F}(s) e^{-sx}\, ds/(2\pi i)$. In (9), the numerator factors ℓ_0, r_0 become the two operators in (16), d_{00}^{-1} supplies a common residual flow, and $(a+b)^{-1}$ remains outside the first-branch integral.

Remark 3.4 (Where the numerator identity is applied). The passage from the numerator factor $\ell_0 = a + X + s_0$ to the operator $\mathcal{L}_a$ requires care, and we state once and for all the stage at which it is used. Under the convention above, the variable conjugate to x_0 is $\sigma_0 = X + s_0$, and for a function F defined only on $x_0 \geq 0$ and extended by zero one has, as distributions on $\mathbb{R}$,

$$(a + \sigma_0)\widehat{F} \;\longleftrightarrow\; \mathcal{L}_a F - F(0^+, x_1, x_2, x_3)\, \delta(x_0), \tag{17}$$

so a jump of F across the coordinate face $x_0 = 0$ would contribute a boundary distribution. We never incur that term, because (21) is proved and used only at the *smooth tensor stage* of Lemma 3.10, where each support weight is a finite sum of products of functions in $C_c^\infty(\mathbb{R})$ together with a smooth radial factor. Such an F is smooth on all of $\mathbb{R}^4$; it has no jump across $x_0 = 0$ (nor across any $x_i = 0$), the second term of (17) vanishes identically, and $\mathcal{L}_a F$ is the classical derivative. The flow map (20) takes values in the closed orthant, so only $x_0 \geq 0$ is ever sampled. The interchange of the $\boldsymbol{\sigma}, \boldsymbol{\tau}, \xi, \psi$ contour integrals with the flow integrals is likewise justified at that stage, by the normal convergence and Schwartz bounds (MP.8a) and (MP.13); the real parts (MP.1) are exactly what makes each representation $1/h = \int_0^\infty e^{-hv}\, dv$ below absolutely convergent. The sharp weight F_S of (15) is reached only afterwards, by the $W^{1,1}$ limit (MP.21)–(MP.21a) in the radial variable and by bounded almost-everywhere convergence in the selector; the one face at which that limit could produce a distribution is the slanted face $|\boldsymbol{x}| = 1$, and Lemma 3.13 shows that $P_3(0) = 0$ removes it. No coordinate face contributes at any stage.

Fix a partial matching M, and let

$$I = \{1, 2, 3\} \setminus V_L(M), \qquad J = \{1, 2, 3\} \setminus V_R(M). \tag{18}$$

Upon expanding the unmatched differences in (11), let $U \subseteq I$ and $V \subseteq J$ denote the vertices for which the one-sided denominators ℓ_i^{-1} and r_j^{-1} were chosen. Introduce nonnegative flow variables

$$w, \qquad z_{ij}\ ((i,j) \in M), \qquad u_i\ (i \in I), \qquad v_j\ (j \in J), \tag{19}$$

and define

$$
\begin{aligned}
x_0 &= w + \sum_{j\in J\setminus V} v_j, & y_0 &= w + \sum_{i\in I\setminus U} u_i,\\
x_i &= \begin{cases} z_{ij}, & (i,j)\in M,\\ u_i, & i\in I,\end{cases} & y_j &= \begin{cases} z_{ij}, & (i,j)\in M,\\ v_j, & j\in J.\end{cases}
\end{aligned}
\tag{20}
$$

Proposition 3.5 (Flow formula and Q-operator)**.** *Before division by $a+b$, the first branch is*

$$
\begin{aligned}
\mathcal{H}_{a,b}(F,G) = \sum_{M\in\mathcal{M}_{3,3}} \sum_{U\subseteq I} \sum_{V\subseteq J} (-1)^{|U|+|V|} \int_{\mathbb{R}_+^{7-|M|}} \exp\left(-a\sum_{i\in U} u_i - b\sum_{j\in V} v_j\right)&\\
\times \left(\prod_{(i,j)\in M} z_{ij}\right) \mathcal{L}_a F(\boldsymbol{x}) \mathcal{L}_b G(\boldsymbol{y})\, d\lambda.&
\end{aligned}
\tag{21}
$$

Here $d\lambda$ is Lebesgue measure in all variables in (19); the simplex supports of F, G make the domain compact. The two branches combine as

$$
\mathcal{C}_{a,b}(F,G) = \frac{\mathcal{H}_{a,b}(F,G) - e^{-(a+b)/\theta}\mathcal{H}_{-b,-a}(F,G)}{a+b}, \tag{22}
$$

and the normalized diagonal is

$$
\boxed{\mathcal{D}_Q(F,G) = Q(-\theta\partial_a)Q(-\theta\partial_b)\mathcal{C}_{a,b}(F,G)\big|_{a=b=-\theta R}\,.} \tag{23}
$$

Proof. Use, term by term,

$$
\frac{1}{h} = \int_0^\infty e^{-hv}\, dv, \qquad \frac{1}{h^2} = \int_0^\infty v e^{-hv}\, dv. \tag{24}
$$

A matched denominator d_{ij}^{-2} gives a common coordinate $x_i = y_j = z_{ij}$ and the factor z_{ij}. For an unmatched left vertex, ℓ_i^{-1} gives $x_i = u_i$ and e^{-au_i}, whereas d_{i0}^{-1} gives $x_i = u_i$ and adds u_i to y_0. The right side is identical with left and right exchanged. The denominator d_{00}^{-1} gives the common residual variable $x_0 = y_0 = w$. The minus signs of the one-sided choices give $(-1)^{|U|+|V|}$. These observations prove (21).

The second branch of the twisted second moment replaces (a,b) by $(-b,-a)$ and carries the factor $e^{-(a+b)/\theta}$, giving (22). Finally, because $L = \theta\log T$, the original normalized differential operator in a spectral shift becomes $Q(-\theta\partial_a)$, and similarly on the right. Evaluating at the shifted line gives $a = b = -\theta R$. This is (23). □

3.4 Rectangular Euler factors and the mixed A–D flow

We give the one-sided crosses explicitly. There is a small indexing point which is important here. A component $(\mu * \Lambda^{*q})P/L^q$ has q marked vertices, whereas the classical P_1-component in (LC.9) is μP_1 and has no marked vertex. Thus the four shortened pieces use the rectangular graphs

$$
K_{0,3}, \qquad K_{2,3}, \qquad K_{4,3}, \qquad K_{5,3},
$$

respectively. We put the unrestricted A-side first; exchanging the two sides gives the $K_{3,q}$ notation. A genuine $\mu * \Lambda$ component would use $K_{1,3}$, and the formula below includes that case, but no such component occurs in the mollifier.

For $q \geq 0$, introduce left marker variables $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_q)$, with the empty tuple when $q = 0$, and retain the three right marker variables $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$. At a prime p, put

$$\begin{aligned} F_{p,q}(X;\boldsymbol{\eta}) &= (1 - p^{-\sigma_0}X)\prod_{i=1}^{q} \frac{1 - p^{-\sigma_i}X}{1 - p^{-\sigma_i-\eta_i}X} = \sum_{r\geq 0} A_{p,q}(r;\boldsymbol{\eta})X^r, \\ F^R_{p,3}(X;\boldsymbol{\nu}) &= (1 - p^{-\tau_0}X)\prod_{j=1}^{3} \frac{1 - p^{-\tau_j}X}{1 - p^{-\tau_j-\nu_j}X} = \sum_{s\geq 0} B_{p,3}(s;\boldsymbol{\nu})X^s. \end{aligned} \tag{25}$$

The exact first-branch local diagonal is

$$E_p^{(q,3)} = \sum_{r,s\geq 0} A_{p,q}(r)B_{p,3}(s)p^{(1+\alpha+\beta)\min(r,s)-(1+\alpha+\xi)r-(1+\beta+\psi)s}. \tag{26}$$

Its squarefree local factor is

$$E_{p,\mathrm{sf}}^{(q,3)} = 1 + \frac{L_{p,q}}{p^{\mathfrak{a}}} + \frac{R_{p,3}}{p^{\mathfrak{b}}} + \frac{L_{p,q}R_{p,3}}{p^{\mathfrak{c}}}, \tag{27}$$

where, throughout this subsection,

$$\mathfrak{a} = 1 + \alpha + \xi, \qquad \mathfrak{b} = 1 + \beta + \psi, \qquad \mathfrak{c} = 1 + \xi + \psi,$$

the same shifted exponents recorded in (65) below, and

$$\begin{aligned} L_{p,q} &= -p^{-\sigma_0} + \sum_{i=1}^{q}(p^{-\sigma_i-\eta_i} - p^{-\sigma_i}), \\ R_{p,3} &= -p^{-\tau_0} + \sum_{j=1}^{3}(p^{-\tau_j-\nu_j} - p^{-\tau_j}), \end{aligned} \tag{28}$$

The polar factor which matches every prime-degree-one term in (27) is

$$\mathcal{Z}_{q,3}^{\mathrm{pol}} = \mathcal{Z}_L^{(q)}\mathcal{Z}_R^{(3)}\mathcal{Z}_{LR}^{(q,3)}, \tag{29}$$

with

$$\begin{aligned} \mathcal{Z}_L^{(q)} &= \frac{1}{\zeta(\mathfrak{a}+\sigma_0)}\prod_{i=1}^{q}\frac{\zeta(\mathfrak{a}+\sigma_i+\eta_i)}{\zeta(\mathfrak{a}+\sigma_i)}, \\ \mathcal{Z}_R^{(3)} &= \frac{1}{\zeta(\mathfrak{b}+\tau_0)}\prod_{j=1}^{3}\frac{\zeta(\mathfrak{b}+\tau_j+\nu_j)}{\zeta(\mathfrak{b}+\tau_j)}, \end{aligned} \tag{30}$$

and

$$\begin{aligned} \mathcal{Z}_{LR}^{(q,3)} &= \zeta(\mathfrak{c}+\sigma_0+\tau_0) \\ &\quad\times \prod_{j=1}^{3}\frac{\zeta(\mathfrak{c}+\sigma_0+\tau_j)}{\zeta(\mathfrak{c}+\sigma_0+\tau_j+\nu_j)} \\ &\quad\times \prod_{i=1}^{q}\frac{\zeta(\mathfrak{c}+\sigma_i+\tau_0)}{\zeta(\mathfrak{c}+\sigma_i+\eta_i+\tau_0)} \\ &\quad\times \prod_{i=1}^{q}\prod_{j=1}^{3}\frac{\zeta(\mathfrak{c}+\sigma_i+\eta_i+\tau_j+\nu_j)\zeta(\mathfrak{c}+\sigma_i+\tau_j)}{\zeta(\mathfrak{c}+\sigma_i+\eta_i+\tau_j)\zeta(\mathfrak{c}+\sigma_i+\tau_j+\nu_j)}. \end{aligned} \tag{31}$$

Empty products in these formulas are one.

Lemma 3.6 (The two rectangular local quotients)**.** *For each fixed integer q with $0 \le q \le 5$, in a sufficiently small fixed polydisc and for every fixed multi-index γ in the support, outer, spectral, and marker variables,*

$$\partial^\gamma \left(\frac{E_p^{(q,3)}}{E_{p,\mathrm{sf}}^{(q,3)}} - 1 \right) \ll_{q,\gamma} \frac{(\log p)^{|\gamma|}}{p^{2-\varepsilon}}, \qquad \partial^\gamma \left(\frac{E_{p,\mathrm{sf}}^{(q,3)}}{(\mathcal{Z}_{q,3}^{\mathrm{pol}})_p} - 1 \right) \ll_{q,\gamma} \frac{(\log p)^{|\gamma|}}{p^{2-\varepsilon}}. \tag{32}$$

Define

$$H_{\mathrm{usf}}^{(q,3)} := \prod_p \frac{E_p^{(q,3)}}{E_{p,\mathrm{sf}}^{(q,3)}}, \qquad H_{\mathrm{sfp}}^{(q,3)} := \prod_p \frac{E_{p,\mathrm{sf}}^{(q,3)}}{(\mathcal{Z}_{q,3}^{\mathrm{pol}})_p}.$$

Both Euler-product quotients are therefore holomorphic, together with all the fixed derivatives which occur here. At zero markers the first quotient is one. At zero markers and zero support shifts, and at $(\xi, \psi) = (\beta, \alpha)$, the second quotient is also one.

Proof. At zero markers, (25) gives $F_{p,q}(X; \mathbf{0}) = 1 - p^{-\sigma_0} X$, independently of q. Thus every coefficient of X^r, $r \ge 2$, vanishes at the marker origin. Away from the origin, every term removed in passing from (26) to (27) has $\max(r,s) \ge 2$. Its exponent of p is at most $-2 + O(\varepsilon)$; a fixed differentiation in any of the variables listed in the statement supplies only a fixed power of $\log p$. This proves the first bound in (32), including the assertion at the marker origin.

Equations (28) and (31) show term by term that the logarithm of $\mathcal{Z}_{q,3}^{\mathrm{pol}}$ has exactly the three nonconstant prime-degree-one contributions $L_{p,q} p^{-\mathfrak{a}}$, $R_{p,3} p^{-\mathfrak{b}}$, and $L_{p,q} R_{p,3} p^{-\mathfrak{c}}$. Hence the second local quotient starts in prime degree two, which proves the second bound. At the stated base point put $z_p = p^{-1-\alpha-\beta}$. Then $L_{p,q} = R_{p,3} = -1$, so $E_{p,\mathrm{sf}}^{(q,3)} = 1 - z_p$. The two one-sided polar factors contribute $(1 - z_p)^2$ and the cross factor contributes $(1 - z_p)^{-1}$, also giving $1 - z_p$. This proves the last assertion prime by prime. Absolute convergence of the two bounds proves all the claimed holomorphy statements. In particular, after the $q + 3$ marker derivatives and normalization by L^{q+3}, a derivative falling on either quotient loses at least one polar logarithm and is $o(1)$, exactly by the polar-logarithm count proved in Proposition 3.9 below. □

Scale the variables as in (7), now with $1 \le i \le q$ on the left and $1 \le j \le 3$ on the right. The leading poles in (29) give

$$\frac{\ell_0 r_0}{(a+b) d_{00}} G_{q,3}(\boldsymbol{e}, \boldsymbol{f}), \tag{33}$$

where

$$\begin{aligned} G_{q,3} = \prod_{i=1}^{q} \frac{\ell_i}{\ell_i + e_i} \prod_{j=1}^{3} \frac{r_j}{r_j + f_j} \prod_{i=1}^{q} \frac{d_{i0} + e_i}{d_{i0}} \prod_{j=1}^{3} \frac{d_{0j} + f_j}{d_{0j}} \\ \times \prod_{i=1}^{q} \prod_{j=1}^{3} \frac{(d_{ij} + e_i)(d_{ij} + f_j)}{d_{ij}(d_{ij} + e_i + f_j)}. \end{aligned} \tag{34}$$

Let $\mathcal{M}_{q,3}$ be the partial matchings between $[q] = \{1, \dots, q\}$ and $[3]$. For such an M, define

$$\begin{aligned} V_L(M) &= \{ i \in [q] : (i,j) \in M \text{ for some } j \}, \\ V_R(M) &= \{ j \in [3] : (i,j) \in M \text{ for some } i \}. \end{aligned} \tag{35}$$

Then the required squarefree coefficient is the concrete finite sum

$$[e_1\cdots e_q f_1 f_2 f_3]G_{q,3} = \sum_{M\in\mathcal{M}_{q,3}} \prod_{(i,j)\in M} \frac{1}{d_{ij}^2} \times \prod_{i\notin V_L(M)} \left(\frac{1}{d_{i0}} - \frac{1}{\ell_i}\right) \prod_{j\notin V_R(M)} \left(\frac{1}{d_{0j}} - \frac{1}{r_j}\right). \tag{36}$$

Indeed, the identity (14) gives either no markers or the pair $e_i f_j/d_{ij}^2$, and every remaining marker must come from its one-sided difference. Hence selected cross factors are exactly a partial matching, which proves (36) without an appeal to the 3-by-3 case. The number of summands is

$$\#\mathcal{M}_{q,3} = \sum_{j=0}^{\min(q,3)} \binom{q}{j}\binom{3}{j} j!. \tag{37}$$

For $q = 0, 1, 2, 3, 4, 5$, these numbers are respectively $1, 4, 13, 34, 73, 136$.

For $M \in \mathcal{M}_{q,3}$, put $I = [q] \setminus V_L(M)$ and $J = [3] \setminus V_R(M)$. If $U \subseteq I$ and $V \subseteq J$, introduce $w, z_{ij}, u_i, v_j \geq 0$ and set

$$\begin{aligned} x_0 &= w + \sum_{j\in J\setminus V} v_j, & y_0 &= w + \sum_{i\in I\setminus U} u_i, \\ x_i &= \begin{cases} z_{ij}, & (i,j)\in M, \\ u_i, & i \in I, \end{cases} & y_j &= \begin{cases} z_{ij}, & (i,j)\in M, \\ v_j, & j \in J. \end{cases} \end{aligned} \tag{38}$$

Termwise inverse Laplace transformation of (36) gives

$$\begin{aligned} \mathcal{H}_{a,b}^{(q,3)}(F,G) = \sum_M \sum_{U\subseteq I} \sum_{V\subseteq J} (-1)^{|U|+|V|} \int_{\mathbb{R}_+^{q+4-|M|}} \\ e^{-a\sum_{i\in U} u_i - b\sum_{j\in V} v_j} \prod_{(i,j)\in M} z_{ij}(a-\partial_{x_0})F(\boldsymbol{x}) \\ \times (b-\partial_{y_0})G(\boldsymbol{y})\, d\lambda. \end{aligned} \tag{39}$$

There are $1 + |M| + (q - |M|) + (3 - |M|) = q + 4 - |M|$ integration variables. Formula (39) follows directly from $d^{-1} = \int_0^\infty e^{-dv}\, dv$ and $d^{-2} = \int_0^\infty v e^{-dv}\, dv$: a matched edge gives the common mass z_{ij} and its factor z_{ij}; an unmatched one-sided choice gives the corresponding exponential; and a common residual choice adds the flow to the opposite residual coordinate. This proves the rectangular flow formula, including its signs and dimension.

The marker-sign convention. The Taylor coefficient of (25) is

$$[\eta_1\cdots\eta_q] \frac{1}{\zeta(z+\sigma_0)} \prod_{i=1}^{q} \frac{\zeta(z+\sigma_i+\eta_i)}{\zeta(z+\sigma_i)} = (-1)^q (\mu * \Lambda^{*q})_{\boldsymbol{\sigma}}. \tag{40}$$

Consequently (39) represents the usual Feng-signed component $(-1)^q \mu * \Lambda^{*q}$. If the arithmetic coefficient is instead written with the positive convolution $\mu * \Lambda^{*q}$, the entire q-side of (39) must be multiplied by $(-1)^q$. This scalar cannot be discarded in a mixed q-against-3 pairing (it happened to cancel in the 3-against-3 pairing).

Every formula in this paper uses the Feng-signed convention (40) on both sides, without exception. The mollifier coefficient (LC.9) and the support-weighted coefficient (1) are already written in that convention: both carry the factor $(-1)^3 = -1$ on the degree-three component, so (LC.9) reads equivalently

$$\begin{aligned} a(n) = \mu(n)\widetilde{P}_1(x_n) + \sum_{k\in\{2,4,5\}} \frac{(-1)^k(\mu * \Lambda^{*k})(n)}{L^k}\widetilde{P}_k(x_n) \\ - \frac{P_3(x_n)}{L^3} \sum_{mq_1q_2q_3=n} \mu(m)\Lambda(q_1)\Lambda(q_2)\Lambda(q_3)W(t_1,t_2,t_3), \end{aligned} \tag{41}$$

and C_S carries the same minus sign. We stress the point only because the scalar $(-1)^q$ does not cancel in a mixed q-against-3 pairing: an argument written with the positive convolution $\mu * \Lambda^{*q}$ must negate the entire q-side of (39) for every odd q.

We next collapse (39) to the exact four-state fiber used for AD. Define the zero-extended radial polynomials

$$p_3(u) = P_3(u), \qquad p_k(u) = \begin{cases} P_k((u-x_*)/\rho), & u \ge x_*, \\ 0, & u < x_*, \end{cases} \quad (k = 1,2,4,5). \tag{42}$$

Thus p_k is continuous at x_*, because $P_k(0) = 0$. For $k \ge 2$, $r \ge 0$, and a scaled spectral variable a, put

$$J_{k,r}(a,u) = \int_0^u p_k(u-\mu)\mu^r e^{-a\mu}\, d\mu. \tag{43}$$

The integrand is zero for $\mu > u - x_*$ when $k \ne 3$. The contribution of the k-th Feng component to the exposed-count state m is

$$V_m^{[k]}(a,u) = \begin{cases} \binom{k}{m}\dfrac{(-1)^{k-m}}{(k-m-2)!}J_{k,k-m-2}(a,u), & 0 \le m \le k-2, \\ -kp_k(u), & m = k-1, \\ ap_k(u) + p_k'(u), & m = k, \\ 0, & \text{otherwise}, \end{cases} \qquad (k \ge 2), \tag{44}$$

whereas the classical component has

$$V_0^{[1]}(a,u) = ap_1(u) + p_1'(u), \qquad V_m^{[1]}(a,u) = 0 \quad (m > 0). \tag{45}$$

For reference, (44) is the following explicit finite table (blank entries are zero):

	V_0	V_1	V_2	V_3	V_4	V_5
P_1	$ap_1 + p_1'$					
P_2	$J_{2,0}$	$-2p_2$	$ap_2 + p_2'$			
P_3	$-J_{3,1}$	$3J_{3,0}$	$-3p_3$	$ap_3 + p_3'$		
P_4	$\frac{1}{2}J_{4,2}$	$-4J_{4,1}$	$6J_{4,0}$	$-4p_4$	$ap_4 + p_4'$	
P_5	$-\frac{1}{6}J_{5,3}$	$\frac{5}{2}J_{5,2}$	$-10J_{5,1}$	$10J_{5,0}$	$-5p_5$	$ap_5 + p_5'$

(46)

Here every $J_{k,r}$ in a row has the arguments (a,u). The full unrestricted jet is

$$V_m^A(a,u) = \sum_{k=1}^{5} V_m^{[k]}(a,u), \qquad 0 \le m \le 5. \tag{47}$$

For completeness, (44) follows from the rectangular flow, rather than being an imported identity. Fix a $q = k$ component and suppose exactly m of its k marked vertices feed the opposite side. There are $\binom{k}{m}$ choices and the $k-m$ hidden vertices give the sign $(-1)^{k-m}$. If $r = k-m \geq 1$, marginalizing their individual masses at total μ gives

$$(-1)^r \binom{k}{m} \int_0^u \frac{\mu^{r-1}}{(r-1)!} e^{-a\mu} \{ap_k(u-\mu) + p_k'(u-\mu)\}\, d\mu. \tag{48}$$

Since

$$\frac{d}{d\mu}\{p_k(u-\mu)e^{-a\mu}\} = -e^{-a\mu}\{ap_k(u-\mu) + p_k'(u-\mu)\}, \tag{49}$$

integration by parts gives $J_{k,r-2}/(r-2)!$ when $r \geq 2$, and $p_k(u)$ when $r = 1$. The endpoint at the other end is zero because the zero extension in (42) is continuous and vanishes at its radial boundary. When $r = 0$, no hidden integral is present and the value is $ap_k + p_k'$. Multiplication by $(-1)^r\binom{k}{m}$ proves all three lines of (44). The P_1 coefficient has $q = 0$, which proves (45) separately.

Let $\mathfrak{D}$ denote the deleted marked-coordinate set of Section 4 and write its coordinates as $\boldsymbol{x} = (x_1, x_2, x_3)$. For $0 \leq n \leq 3$, put

$$E_n = \sum_{i=1}^{n} x_i, \qquad H_n = \sum_{i=n+1}^{3} x_i, \qquad U_n = 1 - H_n, \qquad y_n = 1 - t - H_n. \tag{50}$$

For $0 \leq m \leq 5$, define the exact matching density

$$K_{m,n}(\boldsymbol{x}, t) = \sum_{j=0}^{\min(m,n)} (m)_j \binom{n}{j} \left(\prod_{i=1}^{j} x_i\right) \frac{(t-E_n)^{m-j}}{(m-j)!}, \qquad (m)_j = \frac{m!}{(m-j)!}. \tag{51}$$

Then the first-branch mixed flow, before division by $a + b$, is

$$\boxed{\begin{aligned}\mathcal{H}_{a,b}(A, D) = \sum_{n=0}^{3} (-1)^{3-n} \binom{3}{n} \int_{\mathfrak{D}} \int_{E_n}^{U_n} \sum_{m=0}^{5} K_{m,n}(\boldsymbol{x}, t) V_m^A(a, 1-t) \\ \times e^{-bH_n}\{bP_3(y_n) + P_3'(y_n)\}\, dt\, d\boldsymbol{x}.\end{aligned}} \tag{52}$$

This equation is in the Feng marker convention of (40); the convention scalars described after that equation must be inserted if a different arithmetic sign convention is retained.

We prove (52) directly. On the selected side, let n be the number of marked vertices which feed the unrestricted side. The other $3 - n$ one-sided choices give the sign $(-1)^{3-n}$, and symmetry of $\mathfrak{D}$ permits us to take the first n vertices and multiply by $\binom{3}{n}$. Their total mass is E_n; the hidden selected mass is H_n. If exactly j of these n vertices are matched to the m exposed unrestricted vertices, there are $\binom{n}{j}(m)_j$ injections. The j squared denominators give $\prod_{i=1}^{j} x_i$. The remaining $m - j$ exposed unrestricted variables, together with the residual variable w, have total $t - E_n$. Eliminating w and integrating their simplex gives exactly $(t - E_n)^{m-j}/(m - j)!$. This proves (51).

The selected support condition is $t + H_n \leq 1$, while nonnegativity of the residual simplex is $t \geq E_n$; hence the vertical interval is precisely $[E_n, U_n]$. Its radial polynomial has argument $y_n = 1 - t - H_n$, and its one-sided exponential is e^{-bH_n}, giving the last line of (52). All hidden unrestricted variables and their signs have already been summed in (47). This accounts for every matching, one-sided choice, and flow variable in (39), proving the reduction.

The four values of n are the four vertical-fiber states

$$\begin{array}{c|c|c|c} n & E_n & H_n & [E_n, U_n] \\ \hline 0 & 0 & x_1+x_2+x_3 & [0, 1-x_1-x_2-x_3] \\ 1 & x_1 & x_2+x_3 & [x_1, 1-x_2-x_3] \\ 2 & x_1+x_2 & x_3 & [x_1+x_2, 1-x_3] \\ 3 & x_1+x_2+x_3 & 0 & [x_1+x_2+x_3, 1] \end{array} \tag{53}$$

There is a second four-way split, namely the exact cutoff partition inside each n-state. Recall $x_* = 3/203$ and $\rho = 1 - x_* = 200/203$. Since the shortened pieces vanish when $1 - t < x_*$, their only moving breakpoint is $t = \rho$. The interval $[E_n, U_n]$ is the disjoint union, up to endpoints of measure zero, of the following four possibilities:

$$\begin{array}{c|c|c|c} \text{certificate name} & \text{condition} & \text{polynomial branch} & t\text{-interval} \\ \hline \text{lowA} & H_n \geq x_* & \text{low} & [E_n, U_n] \\ \text{lowB} & H_n \leq x_*,\ E_n \leq \rho & \text{low} & [E_n, \rho] \\ \text{highA} & H_n \leq x_*,\ E_n \leq \rho & \text{high} & [\rho, U_n] \\ \text{highB} & E_n \geq \rho & \text{high} & [E_n, U_n]. \end{array} \tag{54}$$

Indeed $H_n \geq x_*$ implies $U_n \leq \rho$, whereas $H_n \leq x_*$ implies $U_n \geq \rho$; in the latter case the lower endpoint is either below or above ρ. Also $E_n \geq \rho$ and $E_n + H_n \leq 1$ imply $H_n \leq x_*$. Thus (54) is exhaustive and has no positive-measure overlap. We name the four regions lowA, lowB, highA, highB as in (54); in particular highB is nonempty and may not be omitted.

Finally, the Taylor jets used by the exact compiler are visible directly from (52). Write $a = \theta(c + ds)$. Then

$$V^{c,d}_{m,r}(u) = [s^r] V^A_m(\theta(c + ds), u), \tag{55}$$

and, with $\lambda = -\theta d H_n$ and $B_c = P_3'(y_n) + \theta c P_3(y_n)$,

$$\begin{aligned} \Delta^{c,d}_{n,r}(\boldsymbol{x}, t) &= [s^r] e^{-\theta(c+ds)H_n} \{P_3'(y_n) + \theta(c + ds)P_3(y_n)\} \\ &= e^{-\theta c H_n} \left\{ B_c \frac{\lambda^r}{r!} + \mathbf{1}_{r\geq 1} \theta d P_3(y_n) \frac{\lambda^{r-1}}{(r-1)!} \right\}. \end{aligned} \tag{56}$$

The positive branch uses $(c, d) = (-R, 1)$ and the reflected branch uses $(c, d) = (R, -1)$. The symbolic antiderivative in t, followed by the four regions in (54), therefore evaluates (52), not a squarefree proxy. The two AFE branches and the Q-operators are then combined by (22)–(23).

In particular, the P_3-against-P_3 part of AD uses $(F, G) = (F_1, F_{S_D})$, where F_1 means F_S with $S \equiv 1$, and DD uses $(F, G) = (F_{S_D}, F_{S_D})$. Equations (26)–(56) supply the previously missing derivation for the P_1, P_2, P_4, P_5 crosses with D.

We now justify the polar coefficient from the exact Euler product. It is enough first to take S to be a smooth tensor weight. Mellin inversion then assigns separate shifts to the residual Möbius atom and to each marked atom. A finite tensor partition handles a general smooth selector, and Lemma 3.13 below justifies the sharp limiting selector.

3.5 Euler-product justification and the two holomorphic quotients

Let $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ be the four left Mellin shifts and introduce marker variables $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$. The exact one-side Dirichlet series is

$$\frac{1}{\zeta(z + \sigma_0)} \prod_{i=1}^{3} \frac{\zeta(z + \sigma_i + \eta_i)}{\zeta(z + \sigma_i)}. \tag{57}$$

Indeed, applying $\partial_{\eta_1}\partial_{\eta_2}\partial_{\eta_3}$ at $\boldsymbol{\eta}=0$ gives the signed labelled convolution $(-1)^3\mu*\Lambda*\Lambda*\Lambda$ used in (1). At a prime p, introduce a bookkeeping variable X and write

$$F_p(X;\boldsymbol{\eta})=(1-p^{-\sigma_0}X)\prod_{i=1}^{3}\frac{1-p^{-\sigma_i}X}{1-p^{-\sigma_i-\eta_i}X}=\sum_{r\geq 0}A_p(r;\boldsymbol{\eta})X^r. \tag{58}$$

The right side is defined in the same way, with shifts $\boldsymbol{\tau}$, markers $\boldsymbol{\nu}$, and coefficients $B_p(t;\boldsymbol{\nu})$. At zero markers all three zeta ratios cancel:

$$F_p(X;\mathbf{0})=1-p^{-\sigma_0}X, \qquad A_p(r;\mathbf{0})=0 \quad (r\geq 2). \tag{59}$$

For the first branch of the twisted second moment, let α,β be the spectral shifts and ξ,ψ the two outer Perron variables. The exact double local diagonal is

$$E_p(\boldsymbol{\eta},\boldsymbol{\nu})=\sum_{r,t\geq 0}A_p(r;\boldsymbol{\eta})B_p(t;\boldsymbol{\nu})p^{(1+\alpha+\beta)\min(r,t)-(1+\alpha+\xi)r-(1+\beta+\psi)t}. \tag{60}$$

This is the local factor supplied by the arithmetic diagonal in the Pratt–Robles–Zaharescu–Zeindler formula [8, Theorem 4.1]. Its squarefree truncation is the restriction $r,t\in\{0,1\}$:

$$E_p^{\mathrm{sf}}=1+\frac{A_p(1)}{p^{1+\alpha+\xi}}+\frac{B_p(1)}{p^{1+\beta+\psi}}+\frac{A_p(1)B_p(1)}{p^{1+\xi+\psi}}. \tag{61}$$

Lemma 3.7 (Unrestricted versus squarefree)**.** *In a fixed sufficiently small polydisc about the origin, and for every fixed multi-index γ in the support, outer, spectral, and marker variables,*

$$\partial^\gamma\left(\frac{E_p}{E_p^{\mathrm{sf}}}-1\right)\ll_\gamma\frac{(\log p)^{|\gamma|}}{p^{2-\varepsilon}}. \tag{62}$$

Thus

$$H_{\mathrm{usf}}=\prod_p\frac{E_p}{E_p^{\mathrm{sf}}} \tag{63}$$

is holomorphic there, together with every fixed derivative in those variables, and

$$H_{\mathrm{usf}}(\mathbf{0},\mathbf{0})=1. \tag{64}$$

The same assertions hold if only one side is squarefree-truncated.

Proof. Every term omitted from (61) has $\max(r,t)\geq 2$. The exponent in (60) is then at most $-2+O(\varepsilon)$; a marker derivative contributes only a fixed power of $\log p$. The local denominator is bounded away from zero after the polydisc is reduced, which proves (62). The corresponding prime sums converge absolutely, proving holomorphy of (63). Equation (59) says more: at zero markers the unrestricted series has no prime degrees $r\geq 2$, and hence $E_p=E_p^{\mathrm{sf}}$ prime by prime. This proves (64). The one-sided assertion has the same proof. ☐

We next remove the squarefree arithmetic factor from its degree-one polar part. Put

$$\mathfrak{a}=1+\alpha+\xi, \qquad \mathfrak{b}=1+\beta+\psi, \qquad \mathfrak{c}=1+\xi+\psi. \tag{65}$$

The prime-degree-one coefficients in the two one-side series are

$$\begin{aligned} L_p&=-p^{-\sigma_0}+\sum_{i=1}^{3}\left(p^{-\sigma_i-\eta_i}-p^{-\sigma_i}\right),\\ R_p&=-p^{-\tau_0}+\sum_{j=1}^{3}\left(p^{-\tau_j-\nu_j}-p^{-\tau_j}\right). \end{aligned} \tag{66}$$

Consequently

$$E_p^{\mathrm{sf}} = 1 + \frac{L_p}{p^{\mathfrak{a}}} + \frac{R_p}{p^{\mathfrak{b}}} + \frac{L_p R_p}{p^{\mathfrak{c}}}, \tag{67}$$

and its p^{-1}-terms are represented by the following zeta product:

$$\mathcal{Z}^{\mathrm{pol}} = \mathcal{Z}_L \mathcal{Z}_R \mathcal{Z}_{LR}, \tag{68}$$

where

$$\begin{aligned} \mathcal{Z}_L &= \frac{1}{\zeta(\mathfrak{a}+\sigma_0)} \prod_{i=1}^{3} \frac{\zeta(\mathfrak{a}+\sigma_i+\eta_i)}{\zeta(\mathfrak{a}+\sigma_i)}, \\ \mathcal{Z}_R &= \frac{1}{\zeta(\mathfrak{b}+\tau_0)} \prod_{j=1}^{3} \frac{\zeta(\mathfrak{b}+\tau_j+\nu_j)}{\zeta(\mathfrak{b}+\tau_j)} \end{aligned} \tag{69}$$

and

$$\begin{aligned} \mathcal{Z}_{LR} = \zeta(\mathfrak{c}+\sigma_0+\tau_0) & \\ \times \prod_{j=1}^{3} & \frac{\zeta(\mathfrak{c}+\sigma_0+\tau_j)}{\zeta(\mathfrak{c}+\sigma_0+\tau_j+\nu_j)} \\ \times \prod_{i=1}^{3} & \frac{\zeta(\mathfrak{c}+\sigma_i+\tau_0)}{\zeta(\mathfrak{c}+\sigma_i+\eta_i+\tau_0)} \\ \times \prod_{i,j=1}^{3} & \frac{\zeta(\mathfrak{c}+\sigma_i+\eta_i+\tau_j+\nu_j)\zeta(\mathfrak{c}+\sigma_i+\tau_j)}{\zeta(\mathfrak{c}+\sigma_i+\eta_i+\tau_j)\zeta(\mathfrak{c}+\sigma_i+\tau_j+\nu_j)}. \end{aligned} \tag{70}$$

For example, the four summands in the marked–marked part of $L_p R_p$ are

$$\begin{aligned} p^{-\sigma_i-\eta_i-\tau_j-\nu_j} - p^{-\sigma_i-\eta_i-\tau_j} & \\ - p^{-\sigma_i-\tau_j-\nu_j} + p^{-\sigma_i-\tau_j}, & \end{aligned} \tag{71}$$

which are exactly the four logarithmic prime terms of the last quotient in (70).

Lemma 3.8 (Squarefree versus polar)**.** *Let $(\mathcal{Z}^{\mathrm{pol}})_p$ be the local factor of* (68)*. Then*

$$H_{\mathrm{sfp}} = \prod_p \frac{E_p^{\mathrm{sf}}}{(\mathcal{Z}^{\mathrm{pol}})_p} \tag{72}$$

is holomorphic in a fixed smaller polydisc, with all fixed derivatives in the support, outer, spectral, and marker variables, because each local quotient is $1 + O(p^{-2+\varepsilon})$. Moreover,

$$H_{\mathrm{sfp}} = 1 \tag{73}$$

at the common support origin $\boldsymbol{\sigma} = \boldsymbol{\tau} = \boldsymbol{\eta} = \boldsymbol{\nu} = 0$ and the Perron base point $(\xi, \psi) = (\beta, \alpha)$.

Proof. Equations (66)–(70) match every prime term of degree one. Their quotient therefore starts in prime degree two, proving the asserted absolute convergence and holomorphy. At the stated base point put $z_p = p^{-1-\alpha-\beta}$. Then $L_p = R_p = -1$, and (67) gives $E_p^{\mathrm{sf}} = 1 - z_p$. At the same point the local factors in (69) are $1 - z_p$ on each side, while (70) contributes $(1 - z_p)^{-1}$. Hence $(\mathcal{Z}^{\mathrm{pol}})_p = 1 - z_p = E_p^{\mathrm{sf}}$ for every prime, proving (73) prime by prime. □

Proposition 3.9 (The logarithmic loss)**.** *After applying the six marker derivatives and the normalization L^{-6}, the complete leading term of the unrestricted P_3-against-P_3 diagonal is obtained by replacing the exact Euler product with $\mathcal{Z}^{\mathrm{pol}}$. Every term in which a marker derivative falls on either H_{usf} or H_{sfp}, and every marker-free Taylor deviation of H_{sfp} from its base point, is $o(1)$. The assertion is uniform for any fixed finite smooth tensor partition of the support.*

Proof. The six polar marker derivatives produce at most six logarithmic prime sums, hence at most L^6. These six powers cancel the two factors L^{-3} in (1). If at least one marker derivative hits one of the holomorphic quotients, it produces an absolutely convergent prime sum, such as

$$\sum_p \frac{(\log p)^j}{p^{2-\varepsilon}} < \infty, \tag{74}$$

in place of one polar logarithm. The remaining polar factor has logarithmic degree at most five. After normalization the contribution is

$$O\left(\frac{(\log L)^C}{L}\right) = o(1). \tag{75}$$

If no marker derivative hits H_{sfp}, holomorphy and (73) give a first-order bound in the displacement from the base point. Lemma 3.10 below treats the genuinely unbounded support contours: after truncation at scaled height $H = L^\kappa$, the pointwise displacement is $O((1+H)/L)$, while the uniform Schwartz moments of the inverse transforms recover the integrated bound $O(L^{-1}(\log L)^C)$. By (64), H_{usf} is already exactly one at zero markers. The outer residues act on both descriptions in the same way. The normalized, fixed-degree Q-operators become derivatives in scaled spectral variables and cannot restore the lost marker power. The reflected branch is obtained by the same substitution used in the twisted second moment. This proves the proposition. Notice that the argument also covers an unrestricted–squarefree cross, which is the form needed for the mixed AD term. □

Lemma 3.10 (Truncated Mellin–Perron contours)**.** *Let $L = \log Y$, and let*

$$(q, r) \in \{(3,3)\} \cup \{(k,3), (3,k) : 0 \le k \le 5\}.$$

Fix a finite smooth tensor approximation to the left and right support weights. More precisely, before taking any sharp-support limit, assume that each left tensor summand is a product of functions in $C_c^\infty(\mathbb{R})$, evaluated at the atom coordinates $x_i = \log q_i/L$ for $0 \le i \le q$, together with a smooth radial factor $p_\epsilon^{\mathrm{L}}(x_0 + \cdots + x_q)$, where $p_\epsilon^{\mathrm{L}} \in C_c^\infty(\mathbb{R})$. Make the analogous assumption on the right coordinates $y_0, \ldots, y_r$, with radial factor $p_\epsilon^{\mathrm{R}}(y_0 + \cdots + y_r)$. The functions and the number of tensor summands are fixed as $Y \to \infty$.

The atom-support Mellin variables $\boldsymbol{\sigma}, \boldsymbol{\tau}$ and the two outer variables ξ, ψ may then be integrated on unbounded vertical lines, but the following procedure is valid.

Choose fixed real parts for the corresponding scaled variables

$$X = L\xi, \qquad Z = L\psi, \qquad s_i = L\sigma_i, \qquad t_j = L\tau_j$$

so that, for the spectral variables $a = L\alpha$, $b = L\beta$ in the fixed compact set used by the two Q-operators, there is a constant $c_\star > 0$ for which

$$\Re(a + X + s_i) \ge c_\star, \qquad \Re(b + Z + t_j) \ge c_\star, \qquad \Re(X + Z + s_i + t_j) \ge c_\star \tag{MP.1}$$

for every $0 \le i \le q$ and $0 \le j \le r$. Such lines exist because the transforms of the smooth support functions are entire: one may keep the atom lines in a fixed bounded strip and take the *real parts of X, Z sufficiently large. Choose the fixed scaled marker circles to have radius $r_\star$, and take $c_\star > 2r_\star$. Thus every version of the three expressions in* (MP.1) *obtained by adding one or two marker variables still has real part at least $c_\star - 2r_\star > 0$. The same lines can be chosen for the reflected branch.*

Fix $0 < \kappa < 1$ and set

$$\Upsilon = L^\kappa. \tag{MP.2}$$

Truncate every atom-support and outer vertical contour at scaled height Υ. Then:

1. *For every $A > 0$, the difference between the exact normalized diagonal and the diagonal formed with the truncated support contours is $O_A(L^{-A})$.*

2. *On the truncated contours all unscaled support and outer variables are $O(\Upsilon/L) = o(1)$. The Euler quotients H_{usf} and H_{sfp} in the square case, or $H^{(q,3)}_{\mathrm{usf}}$ and $H^{(q,3)}_{\mathrm{sfp}}$ (or their side exchanges) from Lemma 3.6, together with every fixed derivative needed for the $q + r$ marker derivatives and the two fixed-degree Q-operators, are uniformly bounded there. At zero markers, $H_{\mathrm{usf}} = 1$ identically. If $\mathbf{u}$ denotes all support and outer variables and $\mathbf{u}_0$ is the squarefree–polar base point, then*

   $$H_{\mathrm{sfp}}(\mathbf{u}, \mathbf{0}, \mathbf{0}) - 1 = O\left(\frac{1 + \|\mathbf{U} - \mathbf{U}_0\|}{L}\right) = O\left(\frac{1 + \Upsilon}{L}\right), \tag{MP.3}$$

   uniformly on the truncated contours, where $\mathbf{U} = L\mathbf{u}$ and $\mathbf{U}_0 = L\mathbf{u}_0$.

3. *After the $(q + r)$-marker normalization $L^{-(q+r)}$ and integration against the smooth support transforms, a term in which at least one marker derivative hits either quotient is*

   $$O\left(\frac{(\log L)^C}{L}\right), \tag{MP.4}$$

   and the integrated marker-free use of the deviation in (MP.3) *has the same bound. In particular its cruder uniform estimate before support integration is*

   $$O\left(\frac{(1 + \Upsilon)(\log L)^C}{L}\right) = o(1). \tag{MP.5}$$

   These estimates remain true after the fixed Q-operators and for the reflected branch.

Consequently the exact Euler product may be replaced by its polar product under the full, unbounded smooth-support Mellin integrals, both for the 3-by-3 form and for every rectangular q-by-3 or 3-by-q form with $0 \le q \le 5$. The contour truncation may be removed after the polar replacement, and the smooth radial and tensor approximations may subsequently tend to the desired support in the real flow formula.

Proof. We separate four kinds of auxiliary variables. This prevents an unbounded support contour from being confused with a small Cauchy circle.

First, the $q + r$ marker variables $\boldsymbol{\eta}, \boldsymbol{\nu}$ are not Mellin variables. They are differentiated at zero, or equivalently integrated on fixed circles of radius c/L. Second, the spectral variables α, β used by Q lie on fixed circles of radius c/L about the shifted line. Their scaled versions therefore range over a fixed compact set. Third, the variables $\boldsymbol{\sigma}, \boldsymbol{\tau}$ arise from Mellin inversion of the individual atom-support functions. Fourth, ξ, ψ encode the two outer radial cutoffs. Only the third and fourth families have unbounded vertical contours.

For $w \in C_c^\infty(\mathbb{R})$, use the bilateral Laplace convention

$$\widehat{w}(z) = \int_{\mathbb{R}} w(x)e^{-zx}\,dx, \qquad w(x) = \frac{1}{2\pi i}\int_{\Re z=c} \widehat{w}(z)e^{zx}\,dz. \tag{MP.6}$$

Integration by parts gives, for every $N, r \geq 0$,

$$|\partial_z^r \widehat{w}(c+iy)| \leq C_{N,r,w,c}(1+|y|)^{-N}. \tag{MP.7}$$

At $x = \log q/L$, the exponential in (MP.6) is a Dirichlet shift of size z/L. After the harmless sign relabelling used in the definitions of s_i, t_j, this is precisely the atom-support Mellin variable in the Euler product. The smooth radial factor has the same representation, and its exponential splits over $x_0 + \cdots + x_q$ on the left (and $y_0 + \cdots + y_r$ on the right); its transform therefore supplies the corresponding outer variable. For a fixed finite tensor sum, the product transform satisfies a Schwartz bound of the form

$$|\widehat{\mathcal{W}}(\mathbf{c}+i\mathbf{y})| \leq C_N(1+\|\mathbf{y}\|)^{-N} \tag{MP.8}$$

for every N, and the same is true after any fixed number of support or spectral differentiations. In particular, for every fixed D,

$$\int_{\Re \mathbf{z}=\mathbf{c}} (1+\|\Im \mathbf{z}\|)^D |\widehat{\mathcal{W}}(\mathbf{z})|\,|d\Im \mathbf{z}| \leq C_{D,\mathcal{W}}. \tag{MP.8a}$$

The same bound holds with any subset of the contours truncated. Thus a pointwise polynomial bound in the scaled contour heights never incurs a factor equal to the volume of the contour box.

It follows from (MP.8) that truncating one or all support contours at height Υ changes the support weight, in every fixed C^r-seminorm needed here, by $O_N(\Upsilon^{-N})$. This tail is removed before an Euler quotient is introduced. Thus no growth assertion for an Euler quotient on an arbitrarily high vertical line is required.

For completeness, the resulting change in the arithmetic diagonal is also $O_N(L^B\Upsilon^{-N})$ for some fixed B. Indeed, after division by the displayed powers of L, every coefficient of a fixed tensor piece is bounded by a fixed divisor function times its support-weight error. The exact diagonal and each of its finitely many shift derivatives are bounded absolutely using

$$\sum_{d,e \leq Y} \frac{\tau_K(d)\tau_K(e)}{[d,e]} \ll_K L^{B_K}. \tag{MP.9}$$

To see (MP.9), write $d = gr$, $e = gs$, $(r,s) = 1$, and sum $1/(grs)$ with the usual fixed divisor-function majorants. This produces only a fixed power of L. Taking $N > (A+B)/\kappa$ in (MP.8) proves assertion 1.

We now work only on the truncated contours. By (MP.2), every unscaled support or outer variable has modulus

$$O\left(\frac{1+\Upsilon}{L}\right) = o(1). \tag{MP.10}$$

The fixed real parts in (MP.1) keep every polar zeta argument on the prescribed side of its pole, including after the marker-circle perturbations. In particular, for every fixed r,

$$\zeta^{(r)}(1+w) \ll_r |w|^{-r-1} + 1 \ll_r L^{r+1} \tag{MP.11}$$

on these contours. This is the precise source of the powers of L called *polar logarithms*.

Choose once and for all a small polydisc $\mathcal{P}$ in the unscaled support, outer, marker, and spectral variables on which the relevant local factorizations are valid. For $(q,r) = (3,3)$ these define H_{usf}

and H_{sfp}; for $(q,3)$ they are the quotients $H_{\mathrm{usf}}^{(q,3)}$ and $H_{\mathrm{sfp}}^{(q,3)}$ in Lemma 3.6, and the case $(3,q)$ follows by exchanging the sides. In every case their local logarithms, after every fixed derivative ∂^γ, satisfy

$$\partial^\gamma \log H_p = O_\gamma\left(\frac{(\log p)^{|\gamma|}}{p^{2-\varepsilon}}\right). \tag{MP.12}$$

The right side is summable over primes. Hence the Euler products and all fixed derivatives converge normally on a smaller polydisc $\mathcal{P}' \Subset \mathcal{P}$. In the following three displays, H_{usf} and H_{sfp} denote the corresponding square or rectangular quotients. In particular,

$$\sup_{\mathcal{P}'}\left(|\partial^\gamma H_{\mathrm{usf}}| + |\partial^\gamma H_{\mathrm{sfp}}|\right) \le C_\gamma. \tag{MP.13}$$

For sufficiently large L, (MP.10) places the whole truncated contour box in $\mathcal{P}'$. This proves the required uniform quotient bound; it is independent of Υ because the unscaled box is contained in the one fixed polydisc.

At zero markers, the unrestricted one-side local series is linear in the prime bookkeeping variable. Therefore the unrestricted and squarefree double local factors agree prime by prime, for all support and outer variables in $\mathcal{P}'$, and

$$H_{\mathrm{usf}}(\mathbf{u}, \mathbf{0}, \mathbf{0}) = 1. \tag{MP.14}$$

The second quotient equals one at the common support origin and the Perron base point $(\xi, \psi) = (\beta, \alpha)$. The mean-value theorem, (MP.13), and (MP.10) therefore give

$$|H_{\mathrm{sfp}}(\mathbf{u}, \mathbf{0}, \mathbf{0}) - 1| \le C\|\mathbf{u} - \mathbf{u}_0\| \ll \frac{1 + \|\mathbf{U} - \mathbf{U}_0\|}{L} \ll \frac{1+\Upsilon}{L}, \tag{MP.15}$$

which is (MP.3). The straight segment to the base point remains inside $\mathcal{P}'$ after that polydisc is chosen convex.

We may now differentiate the factorization. On the compact truncated contour box, normal convergence and (MP.13) justify every interchange among the Euler product, the $q+r$ marker derivatives, the fixed Q-derivatives, and the contour integrals. It is useful here to write the normalized marker operator in scaled markers $e_i = L\eta_i$, $f_j = L\nu_j$:

$$L^{-(q+r)} \prod_{i=1}^{q} \partial_{\eta_i} \prod_{j=1}^{r} \partial_{\nu_j} = \prod_{i=1}^{q} \partial_{e_i} \prod_{j=1}^{r} \partial_{f_j}. \tag{MP.16}$$

If a scaled marker derivative hits a quotient $H_\bullet$, meaning either of $H_{\mathrm{usf}}, H_{\mathrm{sfp}}$ or their rectangular analogues, the chain rule and (MP.13) give the explicit saving

$$\partial_{e_i} H_\bullet(\mathbf{U}/L, \mathbf{e}/L, \mathbf{f}/L) = L^{-1}(\partial_{\eta_i} H_\bullet)(\mathbf{U}/L, \mathbf{e}/L, \mathbf{f}/L) = O(L^{-1}), \tag{MP.17}$$

and similarly for f_j and for higher fixed derivatives.

The remaining normalized polar factors have only polynomial growth in the scaled contour heights. Indeed, uniformly for $|w| \ll \Upsilon = o(L)$ and $\Re w \ge c_\star - 2r_\star$, the Laurent expansion at one gives

$$\zeta(1 + w/L) = \frac{L}{w} + O(1), \qquad \frac{1}{\zeta(1 + w/L)} = \frac{w}{L} + O\left(\frac{(1+|w|)^2}{L^2}\right), \tag{MP.18}$$

with analogous bounds after every fixed scaled derivative. Consequently the finite polar product, after its displayed powers of L are cancelled, is bounded by $(1 + \|\Im \mathbf{z}\|)^D$ for some fixed D; possible harmless boundary estimates contribute the stated $(\log L)^C$. Equations (MP.8a) and (MP.17) now give (MP.4) without any factor Υ^m from contour volume.

If no marker hits the second quotient, use the first estimate in (MP.3). Its factor $(1 + \|\mathbf{U} - \mathbf{U}_0\|)/L$ is again absorbed by the first Schwartz moment in (MP.8a), giving the integrated bound $O(L^{-1}(\log L)^C)$; taking the supremum first gives the weaker (MP.5). The fixed Q-operators alter only D, the constant, and the exponent C. The reflected branch ranges over another fixed compact set of scaled spectral variables, so the same contour choice and estimates apply.

Combining assertion 1 with (MP.4)–(MP.5) proves the polar replacement for each fixed smooth tensor and radial approximation. The order of limits is important:

$$\begin{aligned}&\text{fix the smooth weights and tensor mesh; choose } \Upsilon = L^\kappa;\\ &L \to \infty; \qquad \text{then remove the smooth approximations.}\end{aligned} \tag{MP.19}$$

There is no separate appeal to quotient bounds after $\Upsilon \to \infty$.

We finally explain the outer Perron variables and removal of the radial smoothing. For the long selected weight, put $d_3 = \deg P_3$ and write $P_3(x) = \sum_{j=1}^{d_3} p_j x^j$. The exact Perron representation is

$$P_3\left(\frac{\log(Y/n)}{L}\right)\mathbf{1}_{\{n\le Y\}} = \frac{1}{2\pi i}\int_{(c/L)}\left(\frac{Y}{n}\right)^z \sum_{j=1}^{d_3}\frac{p_j j!}{L^j z^{j+1}}\,dz. \tag{MP.20}$$

Thus $X = Lz$ is exactly the outer Perron variable of the manuscript. Its scaled kernel is

$$\sum_{j=1}^{d_3}\frac{p_j j!}{X^{j+1}} = O(|X|^{-2}),$$

because $P_3(0) = 0$. We do not require the Euler quotients to be uniform on this entire unbounded line. Instead choose $p_\epsilon \in C_c^\infty(\mathbb{R})$ which agrees with a smooth extension of

$$p(r) = P_3(1-r)\mathbf{1}_{\{0\le r\le 1\}}$$

away from an ϵ-neighborhood of $r = 1$, and such that

$$\|p_\epsilon - p\|_{W^{1,1}([0,\infty))} \longrightarrow 0. \tag{MP.21}$$

Such an approximation exists precisely without a boundary measure at $r = 1$ because $P_3(0) = 0$. Apply the already proved smooth statement to p_ϵ. After the polar inverse-Laplace transform, every flow domain is compact and the only spatial derivative is $a - \partial_{x_0}$. More explicitly, let $\mathcal{H}_\Omega(p_L, p_R)$ denote any one of the finitely many flow integrals on a cell Ω. The polynomial weights and the chosen inward approximants have p and p' uniformly bounded in L^∞. Expansion of the two first-order radial operators and Fubini in the residual variable $w = x_0 = y_0$ therefore give

$$\begin{aligned}&|\mathcal{H}_\Omega(p_L, p_R) - \mathcal{H}_\Omega(\widetilde{p}_L, \widetilde{p}_R)|\\ &\qquad \le C_\Omega\big(\|p_L - \widetilde{p}_L\|_{W^{1,1}} + \|p_R - \widetilde{p}_R\|_{W^{1,1}}\big).\end{aligned} \tag{MP.21a}$$

Indeed, after the remaining compact variables are fixed, every summand is a bounded coefficient times $p_L^{(i)}(w + A)p_R^{(j)}(w + B)$, $i, j \in \{0, 1\}$; use the uniform boundedness of the polynomial radial weights and their smooth approximants and integrate the difference in w. Thus (MP.21) gives convergence for the radial factor and its one required derivative.

Exactly the same argument applies on a rectangular side. Each shortened radial weight has the form

$$p_k(r) = P_k(1 - r/\rho)\mathbf{1}_{\{0\le r\le\rho\}} \qquad (k \in \{1, 2, 4, 5\}),$$

which is (LC.5) written in the variable $r = \log n/L$. Because every $P_k(0) = 0$, p_k is continuous at its right endpoint and belongs to $W^{1,1}$. It therefore admits inward C_c^∞ approximations converging in $W^{1,1}$, just as in (MP.21). Applying the q-by-3 statement first to these fixed smooth approximations and then passing to the $W^{1,1}$-limit proves the contour replacement for every shortened side that occurs here, $q \in \{0, 2, 4, 5\}$. The long $q = 3$ case is (MP.21), and the unused $q = 1$ case is identical for any endpoint-vanishing $W^{1,1}$ radial weight. The normalization is $L^{-(q+3)}$, not $(\rho L)^{-(q+3)}$, in agreement with (LC.9).

Likewise, the marked selector is kept smooth during the Mellin argument. It is independent of x_0, so $a - \partial_{x_0}$ never differentiates a selector face. Bounded almost-everywhere convergence of the smooth selectors on the compact flow cells then removes the selector smoothing. This proves the final assertion and completes the contour justification. $\square$

Corollary 3.11 (Rigorous logarithmic-loss replacement)**.** *Under the fixed-smooth-weight interpretation above, the conclusions of Proposition 3.9 hold with the full unbounded Mellin–Perron inversions. They also hold for every rectangular q-by-3 and 3-by-q quotient of Lemma 3.6, $0 \le q \le 5$, with $q + 3$ marker derivatives and normalization $L^{-(q+3)}$. More explicitly, the term in which a marker derivative hits an Euler quotient is $O(L^{-1}(\log L)^C)$, and the marker-free Taylor deviation of H_{sfp} is $O(L^{-1}(\log L)^C)$ after support integration (and, more crudely, $O(L^{\kappa-1}(\log L)^C)$ before using the Schwartz moment), for any fixed $0 < \kappa < 1$. Both tend to zero before any tensor, radial, or sharp-selector limit is taken.*

3.6 The 286 flows and the thirty symmetry classes

For a matching of size m, there are $\binom{3}{m}^2 m!$ labelled matchings and $3 - m$ unmatched vertices on each side. The choices of U, V in (21) therefore give

$$N_m = \binom{3}{m}^2 m!\, 2^{\,2(3-m)}. \tag{97}$$

Thus

$$(N_0, N_1, N_2, N_3) = (64, 144, 72, 6), \qquad \sum_{m=0}^{3} N_m = 286. \tag{98}$$

The signed terms should remain grouped as in (21); splitting them into unrelated numerical intervals would destroy the cancellation in the two unmatched differences.

For the remainder of this subsection we specialize to DD, so that the two support weights are both F_{S_D}. In particular, the two weights are equal and symmetric in their three marked coordinates. Then all labelled terms with the same

$$m = |M|, \qquad a_0 = |U|, \qquad b_0 = |V| \tag{99}$$

are carried to one another by coordinate permutations. Their common class multiplier is

$$\mu_{m,a_0,b_0} = (-1)^{a_0+b_0} \binom{3}{m}^2 m! \binom{3-m}{a_0}\binom{3-m}{b_0}. \tag{100}$$

The allowed indices are

$$0 \le m \le 3, \qquad 0 \le a_0, b_0 \le 3 - m, \tag{101}$$

so the number of classes is

$$\sum_{m=0}^{3} (4-m)^2 = 30. \tag{102}$$

Lemma 3.12 (Minimum-cap symmetry)**.** *After all flow variables except w have been fixed, define*

$$
\begin{aligned}
C_L &= 1 - \sum_{(i,j)\in M} z_{ij} - \sum_{i\in I} u_i - \sum_{j\in J\setminus V} v_j, \\
C_R &= 1 - \sum_{(i,j)\in M} z_{ij} - \sum_{j\in J} v_j - \sum_{i\in I\setminus U} u_i.
\end{aligned}
\tag{103}
$$

Then $0 \le w \le \min(C_L, C_R)$, and, with $S_U = \sum_{i\in U} u_i$ and $S_V = \sum_{j\in V} v_j$,

$$C_L - C_R = S_V - S_U. \tag{104}$$

In the sum over all ordered pairs (a_0, b_0), it is therefore enough to integrate the branch $C_L \le C_R$ and multiply by two, except when $a_0 = b_0 = 0$, where the factor is one. The canonical left-minimum classes $a_0 = 0 < b_0$ have only null-boundary support and are exactly empty.

Proof. The two inequalities $|\boldsymbol{x}| \le 1$ and $|\boldsymbol{y}| \le 1$, with the maps (20), are exactly $w \le C_L$ and $w \le C_R$. Subtracting the two expressions in (103) gives (104). Exchanging the two sides maps the right-minimum part of class (m, a_0, b_0) measure-preservingly to the left-minimum part of class (m, b_0, a_0). The multiplier (100) is symmetric in a_0, b_0, and every ordered pair occurs. Hence the two minimum branches have equal total contribution after summation.

If $a_0 = b_0 = 0$, then $S_U = S_V = 0$ and the two caps agree identically; there is one branch rather than two. If $a_0 = 0 < b_0$, the condition $C_L \le C_R$ reads $S_V \le 0$. Nonnegativity of the flow variables forces $v_j = 0$ for every $j \in V$, a set of Lebesgue measure zero. This proves the last assertion. There are $3 + 2 + 1 = 6$ such empty classes, for matching sizes $0, 1, 2$. □

3.7 Absence of selector-face distributions

Lemma 3.13 (Marked faces and the radial endpoint)**.** *Suppose that $S = S(x_1, x_2, x_3)$ is bounded and piecewise smooth, with jumps only across finitely many marked-coordinate affine hyperplanes. If $P_3(0) = 0$, then $\mathcal{L}_a F_S$ contains neither a distribution on a selector face nor a distribution on the radial face $|\boldsymbol{x}| = 1$. More precisely, as a distribution on the open positive orthant,*

$$\mathcal{L}_a F_S = \left(aP_3(1 - |\boldsymbol{x}|) + P_3'(1 - |\boldsymbol{x}|)\right) S(x_1, x_2, x_3)\mathbf{1}_{\{|\boldsymbol{x}|<1\}}. \tag{105}$$

Consequently (21) is an ordinary finite Lebesgue integral, even when both weights are sharp selectors.

Proof. Since S is independent of x_0, its distributional derivative $\partial_{x_0} S$ is zero. Thus none of its jump hyperplanes is struck by the only spatial derivative in (16). For the radial cutoff one has

$$
\begin{aligned}
\partial_{x_0}\left(P_3(1 - |\boldsymbol{x}|)\mathbf{1}_{\{|\boldsymbol{x}|\le 1\}}\right) = \; & -P_3'(1 - |\boldsymbol{x}|)\mathbf{1}_{\{|\boldsymbol{x}|<1\}} \\
& - P_3(0)\,\delta_{\{|\boldsymbol{x}|=1\}}.
\end{aligned}
\tag{106}
$$

The delta coefficient vanishes because $P_3(0) = 0$, and (105) follows. The Q-operators differentiate only a, b, not any support coordinate. Even an all-common matching can therefore produce only a product of bounded selector values, never a square of a surface delta. Finally, the flow domain is compact and all remaining factors are polynomial-exponential functions, so the integral is finite. □

Remark 3.14 (The coordinate faces). Lemma 3.13 is stated on the *open* orthant, and this is not a technicality to be waved away: the zero extension of F_S to $\mathbb{R}^4$ does jump across the coordinate face $x_0 = 0$, where its inner trace $P_3(1 - x_1 - x_2 - x_3)S(x_1, x_2, x_3)$ is not identically zero, so the distributional derivative of that extension carries a term supported on $\{x_0 = 0\}$, exactly as in (17). That term never enters the argument, for the reason recorded in Remark 3.4: the flow formula (21) is derived by transform inversion only for the smooth tensor weights, which are genuinely smooth across every coordinate face, and the sharp weight F_S is reached afterwards as a limit *of the flow integrals themselves* — by (MP.21a) in the radial variable and by dominated convergence in the selector — and not by a second, independent inversion. In that limit $\mathcal{L}_a F_S$ denotes the almost-everywhere defined function (105), whose values on the null set $\bigcup_i \{x_i = 0\}$ are irrelevant to every integral in (21).

If $0 \le S_\varepsilon \le 1$ is a marked-coordinate smoothing with $S_\varepsilon \to S$ almost everywhere, formula (105) gives an integrable bound independent of ε on every compact flow domain. Dominated convergence may therefore be applied to each of the finitely many grouped flows and after the fixed-degree Q-differentiation. This establishes the sharp support-weighted diagonal without a selector-face regularization term or a radial-endpoint correction.

3.8 Assembly of the mean value

We now record, as a single statement, the passage from the analytic input of Section 2 to the bilinear quantity computed in Section 4. This is the only place where the published second-moment results are invoked, and we list explicitly which pair of mollifier components each of them covers.

Proposition 3.15 (Assembly)**.** *Fix $\delta > 0$, a selector buffer $\eta > 0$, a radial buffer $\varepsilon_r > 0$, and a finite tensor mesh $\mathcal{P}$, and let ψ be the corresponding mollifier* (LC.10)*, so that in the notation of* (2)

$$\psi = A_{\rm sh} + C_{W,\eta,\varepsilon_r,\mathcal{P}} \qquad \text{with} \qquad A_{\rm sh} = \sum_{k\in\{1,2,4,5\}} \text{(shortened } P_k \text{ piece)}.$$

Let Φ be a fixed smooth nonnegative majorant of $\mathbf{1}_{[1,2]}$ supported in $[1-\varepsilon_h, 2+\varepsilon_h]$. Write

$$\Psi(s) = \sum_{n\le Y} a(n) n^{-s}; \qquad \psi(\sigma_0 + it) = \Psi\Big(\tfrac12 + it\Big)$$

by the normalization in (LC.10)*. Then the main term $\mathcal{M}_{\alpha,\beta}(\psi,\psi;\Phi)$ of* (SM.3) *describes the mollified second moment with a power saving:*

$$\int_{-\infty}^{\infty} \zeta\Big(\tfrac12+\alpha+it\Big)\,\zeta\Big(\tfrac12+\beta-it\Big)\,\Psi\Big(\tfrac12+it\Big)\,\overline{\Psi\Big(\tfrac12+it\Big)}\Phi\big(\tfrac{t}{T}\big)\,dt = \mathcal{M}_{\alpha,\beta}(\psi,\psi;\Phi) + O(T^{1-\epsilon_0}) \tag{107}$$

uniformly for $|\alpha|, |\beta| \le C/\log T$, and consequently

$$\frac{1}{T}\int_T^{2T} |V(\sigma_0+it)\psi(\sigma_0+it)|^2\,dt \le c_{\delta,\eta,\varepsilon_r,\mathcal{P},\varepsilon_h} + o(1), \qquad c_{\delta,\eta,\varepsilon_r,\mathcal{P},\varepsilon_h} = \|\Phi\|_1\,\mathcal{D}_Q(\psi,\psi), \tag{108}$$

with $\mathcal{D}_Q$ as in Definition 3.1. Moreover

$$c_{\delta,\eta,\varepsilon_r,\mathcal{P},\varepsilon_h} \longrightarrow c_{\rm sel} = c_0 - 2AD + DD \tag{109}$$

as $\delta, \eta, \varepsilon_r, \varepsilon_h \downarrow 0$ and the mesh $\mathcal{P}$ is refined, in the order prescribed in Section 2.4.

Proof. Expanding $\psi\overline{\psi}$ gives finitely many pairs, each a product of one component of ψ against the conjugate of another. Every such pair falls into exactly one of three classes.

(i) Short against short. Both factors are among the four shortened pieces. Each has length $Y^\rho = T^{(\vartheta_*-\delta)\rho} < T^{4/7}$ by (LC.5) and $\vartheta_*\rho = 4/7$. Their coefficients are the Feng components μ,

and $\mu * \Lambda^{*k}$ for $k \in \{2,4,5\}$, against one another. This is exactly the twisted second moment of [8, Theorem 4.1], applied at length $T^{4/7-\epsilon}$ to Feng components of degree at most 5; we use it as published and prove nothing about it here. We record this dependence explicitly, since it is the one input of the present paper whose range of validity we do not reprove: *the published short–short asymptotic must cover $\mu * \Lambda^{*k}$ for $0 \le k \le 5$ at length $T^{4/7-\epsilon}$*, which is the setting of [6] as treated in [8].

(ii) Short against selected. One factor is $C_{W,\eta,\varepsilon_r,\mathcal{P}}$, a finite linear combination of smooth product-box pieces of the form (SM.1) of length $T^{29/50-\delta}$; the other is a shortened piece of length $v_0 \le 4/7 - \epsilon < \vartheta$. This is Theorem 2.2 with b the shortened coefficient, together with its final sentence for the opposite orientation.

(iii) Selected against selected. Both factors are $C_{W,\eta,\varepsilon_r,\mathcal{P}}$. This is Theorem 2.2 with $b = a_{\mathrm{sel}}$ and $v_0 = \vartheta$; the hypothesis $v_0 \le \vartheta$ is satisfied with equality, and the proof of that theorem uses $v_0 = \vartheta_*$, which by the monotonicity of (SM.5) in v_0 dominates every case.

Summing the finitely many pairs gives (107), the error being the sum of the finitely many individual power savings, hence $O(T^{1-\epsilon_0})$ for some $\epsilon_0 > 0$ depending on the fixed parameters. Applying the two Q-operators by Cauchy's formula on circles of radius $c/\log T$ costs $(\log T)^{O(1)}$, which the power saving absorbs; evaluating at $\alpha = \beta = \sigma_0 - \frac{1}{2}$, that is at $a = b = -\theta R$, and dividing the smoothed moment by $T\|\Phi\|_1$ gives $\mathcal{D}_Q(\psi,\psi)$ by Definition 3.1. Since $\Phi(t/T) \ge \mathbf{1}_{[T,2T]}(t)$, division by T gives the unweighted upper bound (108) with the factor $\|\Phi\|_1$. Here $\|\Phi\|_1 = 1 + O(\varepsilon_h)$, as described at the end of Section 2.4.

Finally, $\mathcal{D}_Q$ is sesquilinear, so Proposition 3.2 applies to $\psi = A - D$ and gives $\mathcal{D}_Q(\psi,\psi) = c_0 - 2AD + DD$ once the parameters are removed. At the same time $\|\Phi\|_1 \to 1$ as $\varepsilon_h \downarrow 0$. That removal is legitimate in the stated order by Section 2.4 for the selector, radial, tensor, and length parameters and by Corollary 3.11 for the contour truncation, which yields (109). $\square$

4 The exact finite certificate for the 42.1% candidate

This section isolates the finite calculation from the analytic Levinson–Conrey argument. Every terminating decimal in this section is, by definition, the indicated rational number. Floating-point quadrature, quasi-Monte Carlo output, and statistical error bars play no role.

The length and shift parameters are

$$\theta = \frac{29}{50}, \qquad R = \frac{13}{10}, \qquad \theta_s = \frac{4}{7}, \qquad \rho = \frac{\theta_s}{\theta} = \frac{200}{203}, \qquad x_* = 1 - \rho = \frac{3}{203}. \tag{110}$$

The P_3-piece has limiting length T^{θ}. The pieces P_1, P_2, P_4, P_5 have limiting length T^{θ_s}; in the common master coordinate their profiles are

$$\widetilde{P}_k(x) = \begin{cases} P_k((x - x_*)/\rho), & x \ge x_*, \\ 0, & x < x_*, \end{cases} \qquad k = 1,2,4,5. \tag{111}$$

The exact degree-five data are

$$\begin{aligned} P_1(x) &= .2544x + 1.2742x^2 + .5800x^3 - 1.7182x^4 + .6096x^5, \\ P_2(x) &= .6834x + 7.3227x^2 - 22.2752x^3 + 30.4150x^4 - 15.2953x^5, \\ P_3(x) &= 4.2899x - 34.9701x^2 + 98.8882x^3 - 105.1306x^4 + 32.4893x^5, \\ P_4(x) &= -7.3798x + 60.4440x^2 - 123.7661x^3 + 70.1589x^4 - .7105x^5, \\ P_5(x) &= 9.2239x - 40.1893x^2 + 38.6342x^3 - .4583x^4 + .3503x^5. \end{aligned} \tag{112}$$

Thus $P_k(0)=0$ for all k, and $P_1(1)=1$, as exact rational identities. Put

$$\phi_j(z) = L_j(1-2z) - 1,$$

where L_j is the j-th Legendre polynomial. The derivative profile is

$$Q(z) = 1 + .5550\phi_1(z) - .0498\phi_3(z) + .0039\phi_5(z). \tag{113}$$

Equivalently,

$$Q(z) = 1 - \frac{3147}{5000}z - \frac{27}{40}z^2 - \frac{297}{250}z^3 + \frac{2457}{1000}z^4 - \frac{2457}{2500}z^5. \tag{114}$$

In particular $Q(0)=1$ and $Q'(z)=Q'(1-z)$. The nonzero $-2457z^5/2500$ term is retained in every contraction below.

4.1 The marked selector and its exact cells

Let

$$\mathcal{S} = \left[\frac{8}{29}, \frac{13}{29}\right] \cup \left[\frac{16}{29}, \frac{21}{29}\right] \tag{115}$$

and let the deleted marked set be

$$\mathfrak{D} = \left\{ x \in \mathbb{R}^3_{\geq 0} : x_1+x_2+x_3 \leq 1, \quad \sum_{i\in I} x_i \notin \mathcal{S} \text{ for every } \varnothing \neq I \subseteq \{1,2,3\} \right\}. \tag{116}$$

Boundary conventions do not affect any integral. Define the three closed gap intervals

$$B_0 = [0, 8/29], \qquad B_1 = [13/29, 16/29], \qquad B_2 = [21/29, 1].$$

For a word $\nu = \nu_1 \cdots \nu_7 \in \{0,1,2\}^7$, let C_ν be the intersection of the ambient simplex with

$$x_1 \in B_{\nu_1}, \quad x_2 \in B_{\nu_2}, \quad x_1+x_2 \in B_{\nu_3}, \quad x_3 \in B_{\nu_4},$$

$$x_1+x_3 \in B_{\nu_5}, \quad x_2+x_3 \in B_{\nu_6}, \quad x_1+x_2+x_3 \in B_{\nu_7}.$$

Theorem 4.1 (Exact deleted-cell decomposition)**.** *Up to its boundary hyperplanes, $\mathfrak{D}$ is the disjoint union of the following 23 rational tetrahedra C_ν. The middle column lists the four vertices after multiplication by 29; the last column is the unscaled Euclidean volume.*

ν	$29\,\mathrm{Vert}(C_\nu)$	$\mathrm{vol}(C_\nu)$
0000000	$(0,0,0),(0,0,8),(0,8,0),(8,0,0)$	256/73167
0000011	$(0,5,8),(0,8,5),(0,8,8),(3/2,13/2,13/2)$	9/97556
0000101	$(5,0,8),(13/2,3/2,13/2),(8,0,5),(8,0,8)$	9/97556
0001111	$(0,0,13),(0,0,16),(0,3,13),(3,0,13)$	9/48778
0001122	$(0,5,16),(0,8,13),(0,8,16),(3/2,13/2,29/2)$	9/97556
0001212	$(5,0,16),(13/2,3/2,29/2),(8,0,13),(8,0,16)$	9/97556
0002222	$(0,0,21),(0,0,29),(0,8,21),(8,0,21)$	256/73167
0010001	$(5,8,0),(13/2,13/2,3/2),(8,5,0),(8,8,0)$	9/97556
0010112	$(5,8,8),(8,5,8),(8,8,5),(8,8,8)$	9/48778
0011222	$(5,8,16),(13/2,13/2,29/2),(8,5,16),(8,8,13)$	9/97556
0110011	$(0,13,0),(0,13,3),(0,16,0),(3,13,0)$	9/48778

ν	$29\,\mathrm{Vert}(C_\nu)$	$\mathrm{vol}(C_\nu)$
0110022	$(0,13,8),(0,16,5),(0,16,8),(3/2,29/2,13/2)$	$9/97556$
0111122	$(0,13,13),(0,13,16),(0,16,13),(3,13,13)$	$9/48778$
0120012	$(5,16,0),(13/2,29/2,3/2),(8,13,0),(8,16,0)$	$9/97556$
0120122	$(5,16,8),(13/2,29/2,13/2),(8,13,8),(8,16,5)$	$9/97556$
0220022	$(0,21,0),(0,21,8),(0,29,0),(8,21,0)$	$256/73167$
1010101	$(13,0,0),(13,0,3),(13,3,0),(16,0,0)$	$9/48778$
1010202	$(13,0,8),(29/2,3/2,13/2),(16,0,5),(16,0,8)$	$9/97556$
1011212	$(13,0,13),(13,0,16),(13,3,13),(16,0,13)$	$9/48778$
1020102	$(13,8,0),(29/2,13/2,3/2),(16,5,0),(16,8,0)$	$9/97556$
1020212	$(13,8,8),(29/2,13/2,13/2),(16,5,8),(16,8,5)$	$9/97556$
1120112	$(13,13,0),(13,13,3),(13,16,0),(16,13,0)$	$9/48778$
2020202	$(21,0,0),(21,0,8),(21,8,0),(29,0,0)$	$256/73167$

Their total volume is

$$\mathrm{vol}(\mathfrak{D}) = \frac{2399}{146334}, \qquad \frac{\mathrm{vol}(\mathfrak{D})}{1/6} = \frac{2399}{24389}. \tag{117}$$

Proof. Away from the finitely many endpoint hyperplanes, every one of the seven subset sums belongs to exactly one of B_0, B_1, B_2. Hence the 3^7 cells C_ν are disjoint there and cover $\mathfrak{D}$.

For each word, append the inequalities $x_i \geq 0$ and $x_1 + x_2 + x_3 \leq 1$, select every triple of bounding hyperplanes, and solve the resulting 3-by-3 linear system over $\mathbb{Q}$. Retain a solution exactly when it satisfies every inequality. A bounded full-dimensional polytope has a vertex, so a word with no affinely three-dimensional retained vertex set contributes no volume. This finite enumeration leaves precisely the 23 words in the table. Each retained set has exactly the four listed vertices, and direct substitution verifies all defining inequalities. The determinant formula

$$\mathrm{vol}\,\mathrm{conv}(v_0, v_1, v_2, v_3) = \frac{1}{6}\,|\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)|$$

gives the displayed volumes. Exact fraction addition gives 2399/146334. The ambient simplex has volume 1/6, proving the second identity. Thus every step is a finite rational calculation. ☐

4.2 Exact polynomial–exponential integration

Lemma 4.2 (Rational bounds for the exponential)**.** *For $q \in \mathbb{Q}_{\geq 0}$, put $S_K(q) = \sum_{j=0}^{K} q^j/j!$. If $q < K + 2$, then*

$$S_K(q) \leq e^q \leq S_K(q) + \frac{q^{K+1}}{(K+1)!}\left(1 - \frac{q}{K+2}\right)^{-1}. \tag{118}$$

Both endpoints are rational. For $q < 0$, reciprocal endpoint reversal applied to the interval for e^{-q} gives a rational interval for e^q.

Proof. The lower inequality is positivity of the omitted Taylor terms. Beginning with the first omitted term, every successive ratio is at most $q/(K+2) < 1$, so their sum is bounded by the displayed geometric series. For negative q, use $e^q = 1/e^{-q}$. ☐

Theorem 4.3 (Centered rational simplex enclosure)**.** *Let $\Delta = \operatorname{conv}(v_0, \ldots, v_d) \subset \mathbb{R}^d$ have rational vertices and nonzero volume. Let $p \in \mathbb{Q}[x]$ and let L be a rational affine form. Write $x = v_0 + Bu$, $J = |\det B|$, and $p(v_0 + Bu) = \sum_\alpha c_\alpha u^\alpha$. Put*

$$s = \frac{\min_i L(v_i) + \max_i L(v_i)}{2}, \qquad M = \max_i |L(v_i) - s|. \tag{119}$$

For every $N \geq 0$, one obtains an outward rational interval containing $\int_\Delta p(x) e^{L(x)}\,dx$. Its inputs are exact simplex moments and the rational intervals of Lemma 4.2. Before multiplication by the interval for e^s, a valid symmetric remainder is

$$E_N = \frac{J}{d!}\left(\sum_\alpha |c_\alpha|\right) U_M \frac{M^{N+1}}{(N+1)!}, \tag{120}$$

where $U_M \in \mathbb{Q}$ is any certified upper bound for e^M. More generally, the same conclusion and proof hold for any rational shift s_0, with $M_0 = \max_i |L(v_i) - s_0|$. Thus $s_0 = 0$ gives the origin-based production enclosure, while (119) gives the sharper centered enclosure.

Proof. On the standard simplex $\Delta_d = \{u_i \geq 0 : \sum u_i \leq 1\}$,

$$\int_{\Delta_d} u^\alpha\,du = \frac{\prod_i \alpha_i!}{(d + |\alpha|)!}. \tag{121}$$

Consequently the integral of

$$p(v_0 + Bu) \sum_{n=0}^{N} \frac{(L(v_0 + Bu) - s)^n}{n!}$$

is an exact rational number. An affine function attains its extrema over a simplex at vertices; hence $|L - s| \leq M$ throughout Δ. Taylor's theorem gives the pointwise exponential remainder $e^M M^{N+1}/(N+1)!$. Moreover $|\sum c_\alpha u^\alpha| \leq \sum |c_\alpha|$, because $0 \leq u_i \leq 1$ on Δ_d, and $\operatorname{vol}(\Delta_d) = 1/d!$. This proves (120). Finally multiply the centered interval by the rational interval for e^s, taking the minimum and maximum of all four endpoint products. This remains valid when either interval changes sign. $\square$

The two exponent groups occurring in the DD calculation are reflected. The following identity both accelerates the computation and supplies an exact cross-check.

Lemma 4.4 (Reflected Taylor moments)**.** *Let $A(u) = c + \sum_{j=1}^d a_j u_j$. For a monomial u^α, the order N truncations against e^A and e^{-A} are*

$$T_N^{\pm}(\alpha) = \sum_{k_0 + \cdots + k_d \leq N} \frac{(\pm c)^{k_0}}{k_0!} \prod_{j=1}^{d} \frac{(\pm a_j)^{k_j}}{k_j!} \frac{\prod_{j=1}^d (\alpha_j + k_j)!}{(d + |\alpha| + k_1 + \cdots + k_d)!}. \tag{122}$$

Thus both truncations are rational and may be obtained from one spatial convolution by changing the parity of its total degree. If one exponent is L and the other is $2R - L$, their scalar shifts are s and $2R - s$, and Theorem 4.3 supplies independent outward remainders for the two polynomial factors.

Proof. Multiply the one-variable Taylor series for $e^{\pm c}$ and each $e^{\pm a_j u_j}$, discard total degree greater than N, and integrate term by term using (121). Replacing A by $-A$ multiplies a term of total Taylor degree $k_0 + \cdots + k_d$ by its parity sign. The statement about the reflected scalar shift follows from $2R - L = (2R - s) - (L - s)$. $\square$

4.3 Exact FLINT composition and rational projections

The larger primitive classes first form an aggregate polynomial in three variables and then substitute rational affine maps associated with each simplex. The production implementation uses FLINT only for this exact ring operation.

Lemma 4.5 (Exactness of the FLINT substitution)**.** *Suppose*

$$F \in \mathbb{Q}[X_1, \ldots, X_r], \qquad g_i, h \in \mathbb{Q}[u_1, \ldots, u_d].$$

Pass every coefficient to FLINT as the pair of arbitrary-precision integers consisting of its numerator and denominator. Converting the result of

$$h(u)F(g_1(u), \ldots, g_r(u)) \tag{123}$$

back by reading those integer pairs returns the exact coefficient dictionary of (123)*.*

Proof. The map from a reduced Python fraction a/b to the FLINT element $\mathrm{fmpq}(a, b)$ is the identity embedding of $\mathbb{Q}$. FLINT multivariate addition, multiplication, and composition are operations in the polynomial ring over this exact rational field. Reading the numerator and denominator of every output coefficient applies the inverse embedding. Therefore the composite map is the identity on $\mathbb{Q}[u_1, \ldots, u_d]$. □

No endpoint relies solely on this software-level observation. Exact one-cell regressions compare FLINT with an independently materialized sparse power-and-multiply substitution. The aggregate, mixed-radial, and projected families used below returned identical rational endpoints. A separate adversarial test uses signed nonlinear substitutions and a signed multiplier and obtains coefficientwise identity for outputs with 100 and 71 nonzero terms.

Some $m = 0$ and $m = 1$ classes are reduced before triangulation by integrating a fiber which the aggregate integrand does not see.

Lemma 4.6 (Exact rational fiber projection)**.** *Let $P \subset \mathbb{R}^{d+r}$ be a bounded rational polytope and let the integrand be a polynomial–exponential function of the first d coordinates. Partition the projection into rational polytopes on which the active lower and upper fiber faces are fixed. Exact integration of the fiber yields a rational polynomial density on every projected piece, and integration of the projected density equals the original integral.*

Proof. On an active-face piece, every fiber endpoint is a rational affine form. Repeated integration of a polynomial between affine endpoints gives a rational polynomial in the base variables. Fubini's theorem applies because P is bounded and the integrand is continuous. Boundaries between active-face pieces have measure zero. Summing the projected pieces therefore reproduces the original integral exactly. □

The projected implementations verify this lemma internally before using an exponential enclosure: both the unit density and, when $m > 0$, the matched edge density are integrated in the original and projected geometries and compared as exact fractions.

4.4 The thirty DD classes

Let D denote the deleted marked P_3-component. Write $DD = \mathcal{D}_Q(D, D)$ for its normalized diagonal norm. In a partial matching of the three left and three right marked atoms, let m be the matching size and put $r = 3 - m$. Choose a of the unmatched left atoms and b of the unmatched right atoms in the inclusion–exclusion denominators. The signed orbit multiplier is

$$\gamma_{m,a,b} = (-1)^{a+b} \binom{3}{m}^2 m! \binom{r}{a} \binom{r}{b}. \tag{124}$$

Lemma 4.7 (Orbit enumeration)**.** *There are* 34 *partial matchings,* 286 *labelled flows after expanding the unmatched differences, and*

$$\sum_{m=0}^{3} (4-m)^2 = 30 \tag{125}$$

canonical classes (m, a, b)*. Every labelled flow in such a class occurs once in the corresponding labelled expansion. After the labelled terms in that class are identified by symmetry and collected, their common canonical integral is multiplied by* $\gamma_{m,a,b}$ *from* (124)*.*

Proof. For fixed m, choose the m left atoms, the m right atoms, and a bijection between them. This gives $\binom{3}{m}^2 m!$ matchings. Summing over m gives $1+9+18+6 = 34$. Each of the $2r$ unmatched differences has two terms, so the labelled count is

$$1 \cdot 2^6 + 9 \cdot 2^4 + 18 \cdot 2^2 + 6 = 286.$$

For fixed m, symmetry leaves only the cardinalities $0 \le a, b \le r$, hence $(r+1)^2 = (4-m)^2$ classes. There are $\binom{r}{a}\binom{r}{b}$ choices of the two selected subsets, and inclusion–exclusion contributes $(-1)^{a+b}$, proving (124). □

For the canonical representative, write $z_1, \ldots, z_m$ for matched masses, $\ell_1, \ldots, \ell_r$ for left unmatched masses, and $r_1, \ldots, r_r$ for right unmatched masses. Put

$$p = \ell_1 + \cdots + \ell_a, \qquad q = r_1 + \cdots + r_b,$$

and let t be the sum of the matched masses and all unselected unmatched masses. After removing the common residual variable w, the two radial arguments and caps are

$$y_L = 1 - t - p - w, \quad y_R = 1 - t - q - w, \qquad C_L = 1 - t - p, \quad C_R = 1 - t - q. \tag{126}$$

Equivalently $C_L - C_R = q - p$. The canonical branch is $C_L \le C_R$, or $q \le p$.

Lemma 4.8 (Restoration of the second cap)**.** *After summing all ordered* (a, b)*, the right-minimum branch in class* (m, a, b) *is the image, under side exchange, of the left-minimum branch in class* (m, b, a)*. Hence it is correct to retain only the left-minimum branch and multiply by*

$$\beta_{a,b} = \begin{cases} 1, & a = b = 0, \\ 2, & \textit{otherwise}. \end{cases} \tag{127}$$

The six classes $a = 0 < b$ *have zero volume on the canonical branch.*

Proof. Side exchange swaps p, q, the two selector cells, and C_L, C_R, while preserving the symmetric DD integrand. It therefore sends the right-minimum region for (a, b) bijectively to the left-minimum region for (b, a). If $a = b = 0$, then $p = q = 0$ and the caps agree identically, so there is only one region rather than two. Otherwise the two cap half-spaces have distinct coefficient vectors and their equality face is null. Finally, when $a = 0 < b$, the left-minimum inequality is $q \le p = 0$. Nonnegative unmatched masses force $q = 0$, a null face. □

For completeness, the exact contraction used to form the class integrand can be written entirely over $\mathbb{Q}$. If $Q(z) = \sum_{j=0}^{5} q_j z^j$, set

$$C_{ab} = q_a q_b (-1)^{a+b} a!b!, \tag{128}$$

$$H_{ij} = \frac{(-1)^{i+j}\binom{i+j}{i}}{\theta(-2R)^{i+j+1}}, \tag{129}$$

$$K_{ij} = \sum_{a=i}^{5}\sum_{b=j}^{5} C_{ab} H_{a-i,b-j}, \tag{130}$$

$$K^{\mathrm{e}}_{ij} = \sum_{a=i}^{5}\sum_{b=j}^{5} K_{ab} \frac{(-1)^{a-i+b-j}}{(a-i)!(b-j)!}. \tag{131}$$

For a polynomial P, radial argument y, selected sum S, center c, and slope σ, define the rational jet

$$J_n(P; y, S; c, \sigma) = (P'(y) + \theta c P(y)) \frac{(-\theta\sigma S)^n}{n!} + \mathbf{1}_{n\geq 1} \theta\sigma P(y) \frac{(-\theta\sigma S)^{n-1}}{(n-1)!}. \tag{132}$$

The positive and reflected pre-integration polynomials are respectively

$$F_+(w) = \sum_{i,j=0}^{5} K_{ij} J_i(P_3; y_L, p; -R, 1) J_j(P_3; y_R, q; -R, 1), \tag{133}$$

$$F_-(w) = -\sum_{i,j=0}^{5} K^{\mathrm{e}}_{ij} J_i(P_3; y_L, p; R, -1) J_j(P_3; y_R, q; R, -1). \tag{134}$$

Thus the class is obtained by integrating w exactly from 0 to C_L, multiplying by the matched-edge weight $z_1 \cdots z_m$, and integrating

$$\gamma_{m,a,b}\beta_{a,b} z_1 \cdots z_m \left\{ \left(\int_0^{C_L} F_+(w)\, dw \right) e^{\theta R(p+q)} + \left(\int_0^{C_L} F_-(w)\, dw \right) e^{2R-\theta R(p+q)} \right\} \tag{135}$$

over the left-minimum flow polytope for every ordered pair of cells from Theorem 4.1. Equations (131)–(135), followed by Theorem 4.3, are a finite specification of every table entry.

Proposition 4.9 (Completed exact DD enclosure)**.** *All thirty canonical classes are complete, including the six exact zeros. Exact addition in the deliberately widened theorem-arithmetic ledger gives*

$$DD \in [0.000398378663641030,\ 0.000399828267184290]. \tag{136}$$

The independently maintained sharp ledger gives the smaller enclosure

$$DD \in [0.00039837866367365357\,1,\ 0.000399828267142784482],$$

which is contained in (136) *class by class.*

Proof. Every primitive job performs its polynomial algebra, simplex integration, Taylor remainder, and endpoint accumulation over $\mathbb{Q}$, and emits full rational endpoints. The compact repository registry records directed terminating-decimal supersets of those outputs. Full production fractions are retained for the most expensive rows in their checkpoint ledgers; for the remaining rows, the independently checkable provenance is the stated production command and its deterministic rerun, not an unrecorded binary float. The sharp registry parses every recorded decimal as a fraction, verifies that its key set is exactly the 30 canonical keys, and verifies containment in a separately widened arithmetic registry. Exact fraction addition gives (136). The source-to-ledger manifest also checks every canonical key exactly once before any total is formed. $\square$

Table 2: Outward DD class intervals copied from the completed exact ledgers.

m	(a,b)	certified outward interval
3	(0,0)	$[0.0001282927871668291,\ 0.0001282927871668293]$
2	(0,0)	$[0.0002692954015278428,\ 0.0002692954015278430]$
2	(0,1)	$[0, 0]$
2	(1,0)	$[-0.0003500822660166920,\ -0.0003500822660166918]$
2	(1,1)	$[0.0002834918640343479,\ 0.0002834918640343480]$
1	(0,0)	$[0.00012435042261820,\ 0.00012435042261822]$
1	(0,1)	$[0, 0]$
1	(0,2)	$[0, 0]$
1	(1,0)	$[-0.000338126631251,\ -0.000338126627907]$
1	(1,1)	$[0.0005352522186956030,\ 0.0005366988603522671]$
1	(1,2)	$[-0.000186381721789549,\ -0.000186380726341934]$
1	(2,0)	$[0.000120103484140746,\ 0.000120103484700798]$
1	(2,1)	$[-0.000446589964154162,\ -0.000446588154705364]$
1	(2,2)	$[0.000251377423553907,\ 0.000251377576565828]$
0	(0,0)	$[0.00002296919122516975,\ 0.00002296919122516977]$
0	(0,1)	$[0, 0]$
0	(0,2)	$[0, 0]$
0	(0,3)	$[0, 0]$
0	(1,0)	$[-0.00007214473636771182,\ -0.00007214473636771180]$
0	(1,1)	$[0.00015089566427916,\ 0.00015089566427917]$
0	(1,2)	$[-0.00010067543300019742,\ -0.00010067543300019741]$
0	(1,3)	$[0.00002080220121091,\ 0.00002080220121092]$
0	(2,0)	$[0.00002727797882537,\ 0.00002727797882539]$
0	(2,1)	$[-0.00022037247091600925,\ -0.00022037247091600923]$
0	(2,2)	$[0.00026450194190083083,\ 0.00026450194190083084]$
0	(2,3)	$[-0.00007128404478137233,\ -0.00007128404478137232]$
0	(3,0)	$[0.00000314601632746,\ 0.00000314601632747]$
0	(3,1)	$[0.00008233097004273,\ 0.00008233097004274]$
0	(3,2)	$[-0.00015819631672227459,\ -0.00015819631672227458]$
0	(3,3)	$[0.00005814468312351547,\ 0.00005814468312351549]$

4.5 The completed c_0 and AD certificates

Let A be the formal unfiltered hybrid and let D be its deleted marked P_3-part. Put

$$c_0 = \mathcal{D}_Q(A, A), \qquad AD = \mathcal{D}_Q(A, D), \qquad DD = \mathcal{D}_Q(D, D). \tag{137}$$

The selected coefficient is the exact bilinear identity

$$c_{\mathrm{sel}} = c_0 - 2AD + DD. \tag{138}$$

Theorem 4.10 (Unfiltered rational enclosure)**.** *The exact one-variable compiler at Taylor order* 60 *proves the deliberately widened interval*

$$c_0 \in [2.121699912151076,\ 2.121699912151077]. \tag{139}$$

Moreover, exact positive Taylor arithmetic gives

$$e^{7527/10000} - c_0 > 0.001023727921021. \tag{140}$$

Proof. After the short-coordinate split at $\rho = 200/203$, the compiler groups the two approximate-functional-equation branches into 12 low and 6 high polynomial–exponential integrals. It integrates each polynomial coefficient exactly and applies Theorem 4.3; the summed radius is less than $7.824 \cdot 10^{-69}$, which lies strictly inside (139). For the gate, the positive Taylor sum for $e^{7527/10000}$ is a rational lower bound. Subtracting the exact upper endpoint produced by the compiler exceeds the shortened rational budget $1023727921021/10^{15}$, proving (140). □

Theorem 4.11 (Mixed rational enclosure)**.** *The four exact vertical-fiber states have outward intervals*

$$\begin{array}{c|c}
n & \textit{state interval} \\
\hline
0 & [\ \ 0.00389228986250606,\ \ \ 0.00389228986250608] \\
1 & [-0.01129499080788672,\ -0.01129499080788665] \\
2 & [\ \ 0.01096153653472939,\ \ \ 0.01096153653472947] \\
3 & [-0.00375741372497808,\ -0.00375741372497805].
\end{array} \tag{141}$$

Exact addition of the underlying rational endpoints gives

$$AD \in [-0.000198578135629337,\ -0.000198578135629177]. \tag{142}$$

Consequently

$$-2AD < 0.000397156271258674. \tag{143}$$

Proof. For a state n, put $E = \sum_{i<n} x_i$, $H = \sum_{i \ge n} x_i$, and $U = 1 - H$. With $h = 3/203$, the outer fiber $E \le t \le U$ is partitioned, up to null boundaries, into

piece	marked condition	t-interval
lowA	$H \ge h$	$[E, U]$
lowB	$H \le h,\ E \le \rho$	$[E, \rho]$
highA	$H \le h,\ E \le \rho$	$[\rho, U]$
highB	$E \ge \rho$	$[E, U]$.

Indeed $H \ge h$ implies $U \le \rho$, $E \ge \rho$ implies $H \le h$, and the remaining crossing region satisfies $E \le \rho \le U$. On each piece, a grouped term is $p(x, t)e^{c+\gamma t+\delta H}$. If $\gamma \ne 0$, the polynomial

$$B(t) = \sum_{k=0}^{\deg_t p} \frac{(-1)^k \partial_t^k p(t)}{\gamma^{k+1}}$$

satisfies $B' + \gamma B = p$, so the t-integral is its exact endpoint expression. If $\gamma = 0$, use the ordinary polynomial antiderivative. The remaining rational-simplex integrals are enclosed by Theorem 4.3. This yields the four rational checkpoint intervals in (141). The exact-sum parser checks each full endpoint fraction against its displayed outward decimal before adding them; it gives (142). The adverse endpoint in $-2AD$ is $-2AD_{\mathrm{lower}}$, and outward rounding gives (143). $\square$

4.6 Exact closure of the 42.1% inequality

Theorem 4.12 (Exact threshold certificate)**.** *The completed certificate satisfies*

$$c_{\mathrm{sel}} \in [2.122495447085975,\ 2.122496896689520] \quad \textit{and} \quad c_{\mathrm{sel}} < e^{7527/10000}. \tag{144}$$

Consequently

$$1 - \frac{\log c_{\mathrm{sel}}}{R} > \frac{421}{1000}. \tag{145}$$

Proof. The exact widened ledgers give

$$AD \in [-0.00019857813562935, -0.00019857813562915],$$
$$DD \in [0.000398378663641030,\ 0.000399828267184290].$$

Combining these with (139) and (138), with endpoint reversal in the term $-2AD$, gives the interval in (144). Independently, the positive rational Taylor sum through order 80 gives

$$e^{7527/10000} \geq 2.122723640072098.$$

The directed endpoint comparison is

$$2.122496896689520 < 2.122723640072098,$$

with certified rational slack greater than 0.000226743382578292. This proves the strict exponential inequality. Since $(7527/10000)/R = 579/1000$, taking logarithms gives

$$1 - \frac{\log c_{\mathrm{sel}}}{R} > 1 - \frac{579}{1000} = \frac{421}{1000}.$$

It remains to pass from the limiting sharp diagonal to one actual mollifier. Throughout the finite-parameter precursor, the diagonal form $\mathcal{D}_Q$ is parametrized by the fixed endpoint master scale $L_* = (29/50)\log T$; replacing the endpoint support by $T^{29/50-\delta}$ changes the coefficient profile and its support, not this spectral scaling. More explicitly, put $\lambda = ((29/50) - \delta)/(29/50)$. In the fixed variables $r = \log n/L_*$, $u_0 = \log m/L_*$, and $u_i = \log q_i/L_*$ for $1 \leq i \leq 3$, a piece with q von Mangoldt markers and endpoint radial range $c \in \{\rho, 1\}$ has profile

$$\lambda^{-q} P\left(1 - \frac{r}{\lambda c}\right) \mathbf{1}_{\{0 \leq r \leq \lambda c\}}.$$

Every selected tensor argument is u_i/λ, so the selector walls are $\lambda \mathcal{S}_\eta$. These profiles converge to the endpoint profiles in $W^{1,1}$, whereas the identity $-(\log T)^{-1}\partial_\alpha = -(29/50)\partial_a$ used in the assembly remains exact. The strict slack and the continuity statement (109) allow us first to choose fixed positive $\delta, \eta, \varepsilon_r, \varepsilon_h$ and a fixed finite tensor mesh $\mathcal{P}$ for which the constant in (108) is still below $e^{7527/10000}$. These parameters are fixed before $T \to \infty$. Proposition 3.15 then supplies the required mean-square bound with a power-saving error, and Theorem 2.1 proves (145). No estimate uniform in a joint endpoint limit is used. $\square$

4.7 Reproducibility and adversarial checks

The reference reproduction interpreter is CPython 3.12.13, with the package versions pinned in `requirements-rh421.txt`; some historical production logs were made with Anaconda CPython 3.13.9. This distinction does not affect the rational endpoints, but it is recorded for provenance. Create and activate a clean environment, then run the primary checks from the repository root. The repository itself must be on the module path; no workstation temporary directory is part of the certificate:

```
python3.12 -m venv .venv-rh421
source .venv-rh421/bin/activate
python -m pip install -r requirements-rh421.txt
export PYTHONPATH=.
python rh421_release_hash_audit.py
python variational_421_manifest_audit.py
python marked_subset_cells_exact.py
python variational_421_c0_certificate.py --order 60
python variational_421_ad_sum.py --places 18
python rh421_m1_endpoint_audit.py
python rh421_primitive_endpoint_audit.py
python variational_421_adversarial_numerical_audit.py
python variational_421_dd_m1_projected.py --class 11 --cells 1 --order 10 --workers 1 --
    backend flint --regression
python variational_421_dd_outward_ledger.py
python variational_421_final_arithmetic.py
python variational_421_final_numerical_audit.py --recompute-c0
python variational_421_emit_tex_certificate.py
```

The complete command-to-row map for all thirty canonical classes, including the production order and backend for each family, is recorded in `rh421_certificate_manifest.md`. The projected one-cell regression must report

```
projected_FLINT_sparse_endpoint_identity True pieces 1
class 1 1 1 simplices 3 integrals 6
interval_float 0.00042852562129300524 0.0004285257410741603 radius 5.98905775128337e-11
```

Its exact endpoints are

$$\frac{107131405323251310948585771}{250000000000000000000000000000}, \qquad \frac{26782858817135016841357957}{62500000000000000000000000000}.$$

The identity compares FLINT with the independent sparse-dictionary composition on one projected piece; it is not a duplicate full-class run. The endpoint audit must parse the full primitive fractions for both generic matching-one classes and the exceptional matching-two class, recheck their source hashes, and verify directed containment. The manifest and numerical audits must report all 30 canonical `DD` keys, sharp-inside-wide ledger containment, fresh order-60 `c0` containment, and the strict final exponential gate. Only then may the final command emit the macros displayed at the start of this section. A nonzero exit status or a missing key is a failed certificate, not a zero contribution. The release-hash audit is the final archive-integrity check: it verifies the SHA-256 seal only after the proof sources, retained checkpoints, active TEX inputs, and final PDF have been frozen, and it rejects a missing, duplicated, unsafe, or mismatched entry.

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