Irrationality of the 5-adic zeta value $\zeta_5(3)$

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Abstract

We prove that the Kubota–Leopoldt 5-adic zeta value $\zeta_5(3)$ is irrational. Our proof follows the strategy used by Calegari, Dimitrov and Tang for $\zeta_2(5)$: if $\zeta_5(3)$ were rational, the product of the weight-two Eisenstein series of level five with the overconvergent Eisenstein series of weight -2 would be a weight-zero overconvergent modular function with rational q-expansion; Buzzard's analytic-continuation theorem extends it over the whole supersingular annulus of $X_0(5)$, and an adelic arithmetic holonomy bound with row-wise denominator optimization limits the number of $\mathbf{Q}(f)$-linearly independent functions of this kind. The new ingredients are an explicit third-order modular differential equation for the germ, a monodromy proof that its inhomogeneous order is exactly three, and an exact evaluation of the archimedean logarithmic energy I by Jensen's formula on $X_0(5)$. The holonomy bound forces the quotient $(I+3\log 5)/(-\log 2+3\log 5-45/16)$ to be at least four, while a certificate in rational arithmetic bounds this quotient strictly below four; its numerical value is $3.926639729\ldots$.

1 Introduction

We use Calegari's normalization of the Kubota–Leopoldt zeta function [5, §2.2]: $\zeta_5(3)$ is twice the constant term of the overconvergent Eisenstein series at the character $x \mapsto x^{-2}$ of $\mathbf{Z}_5^\times$. Equivalently (see [5, Lemma 2.4] and [7, (5.3)]),

$$\zeta_5(3) = \lim_{\substack{k\in\mathbf{Z}_{>0},\ k\to+\infty \\ k\to -1 \text{ in } \mathbf{Z}_5 \\ k \text{ odd}}} \zeta(1-2k).$$

Thus the weight $2k$ tends to $+\infty$ in the usual sense while tending to -2 in $\mathbf{Z}_5$, so that the arguments $1-2k$ tend to 3; the parity condition $2k \equiv 2 \pmod 4$ keeps the sequence on the component of weight space containing the character $x \mapsto x^{-2}$. The Euler factor $1-5^{2k-1}$ of the p-adic interpolation tends to 1 and has been omitted. This branch specification is essential because the Kubota–Leopoldt zeta function is meromorphic on weight space, which has several components. We regard $\mathbf{Q}$ as a subfield of $\mathbf{Q}_5$; an element of $\mathbf{Q}_5$ is called irrational if it does not lie in $\mathbf{Q}$. For the other primes and positive odd arguments mentioned below, we use the compatible convention $\zeta_p(s) = L_p(s, \omega_p^{1-s})$, where ω_p is the Teichmüller character (of conductor four when $p=2$); see [9, §2.3]. All logarithms of positive real numbers in this paper are natural logarithms.

Theorem 1.1. $\zeta_5(3) \in \mathbf{Q}_5 \setminus \mathbf{Q}$.

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The nonvanishing settles the case $p = 5$, $n = 1$ of [5, Conjecture 2.5].

For Kubota–Leopoldt zeta values at specified positive odd arguments, the first irrationality results were obtained for the small primes. Calegari proved in 2005 that $\zeta_2(3)$ and $\zeta_3(3)$ are irrational, as well as the 2-adic analogue of Catalan's constant, by means of overconvergent modular forms [5]. Beukers subsequently gave alternative continued-fraction proofs of these small-prime zeta-value results and treated further p-adic L-values and p-adic Hurwitz zeta values [1]. For the fixed argument 3, these classical individual irrationality results concern $p = 2, 3$; Theorem 1.1 adds the prime $p = 5$.

At other specified odd arguments, Calegari–Dimitrov–Tang proved the irrationality of the individual value $\zeta_2(5)$ [7, Theorem 5.2]; as they explain there, this was the first result of their collaboration on arithmetic holonomy bounds. A different Apéry-style proof, together with an explicit upper bound for its irrationality measure, was given by Lai–Sprang–Zudilin [12].

There are also substantial simultaneous results. Sprang proved, for every prime p, a p-adic analogue of the Ball–Rivoal theorem: the dimension of the rational span of the branchwise positive p-adic zeta values up to height s grows at least logarithmically with s [14]. For $p = 2$, Lai showed that, for every integer $s \geq 0$, the interval $[s+3, 3s+5]$ contains an odd j for which $\zeta_2(j)$ is irrational [9]. Lai and Sprang later proved that, for every prime p, there is a constant $c_p > 0$ such that, for every $\epsilon > 0$, at least $(c_p - \epsilon)\sqrt{s/\log s}$ of $\zeta_p(3), \zeta_p(5), \ldots, \zeta_p(s)$ are irrational for all sufficiently large positive odd integers s [11]. In particular, infinitely many odd-argument values are irrational for each fixed prime, but this does not identify the value at argument 3.

For explicit finite sets of arguments, Lai–Lupu–Sprang proved in 2025 that, for every $p \geq 5$, there is an odd integer i with

$$3 \leq i \leq p + \frac{p}{\log p} + 5$$

for which $\zeta_p(i)$ is irrational [10, Corollary 1.2]. Their more precise theorem gives the following bounds for several primes: in each column, at least one of $\zeta_p(3), \zeta_p(5), \ldots, \zeta_p(s_p)$ is irrational [10, Theorem 1.1 and Table 1].

Prime p	5	7	11	13	17	19	101
Largest odd argument s_p	13	13	17	19	23	27	119

In particular, for $p = 7$, at least one of

$$\zeta_7(3), \quad \zeta_7(5), \quad \zeta_7(7), \quad \zeta_7(9), \quad \zeta_7(11), \quad \zeta_7(13)$$

is irrational. This is sharper than the endpoint 15 obtained from the simpler bound above, but it does not establish the irrationality of $\zeta_7(3)$ itself. More generally, these simultaneous results do not settle the individual value $\zeta_p(3)$ for $p \geq 5$. To the author's knowledge, the individual irrationality question remains open for every prime $p \geq 7$, including $7, 11, 13, 17$, and 19.

For $p = 5$, Calegari observed that his direct level-five approximants yield the exponent

$$\frac{3}{3/2 + 3/\log 5} = 0.891794\ldots < 1,$$

and hence fall short of the irrationality criterion [5, §3.2]. Theorem 1.1 supplies the missing individual conclusion for $p = 5$.

Strategy and relation to earlier work

Our proof is an implementation, for $p = 5$, of the strategy that Calegari, Dimitrov and Tang used to prove the irrationality of $\zeta_2(5)$ [7, §5.2]. The strategy has three components. First, following

Calegari [5], one considers the product H of the classical weight-two Eisenstein series of level p with the overconvergent Eisenstein series of weight -2. If $\zeta_p(3)$ is rational, H is a weight-zero overconvergent modular function with rational q-expansion, and Buzzard's analytic-continuation theorem [4, Theorem 5.2] extends it over the entire supersingular locus of $X_0(p)$, which supplies a lower bound for its p-adic radius of convergence in the genus-zero cases. Second, at level five, H satisfies an inhomogeneous linear differential equation of exact order three over $\mathbf{Q}(f)$, giving the linearly independent holonomic germs $1, H, \delta H, \delta^2 H$ with controlled denominators. Third, the adelic arithmetic holonomy bound [7, Theorem 2.6], with the row-wise denominator invariant $\tau(\mathbf{b})$ of [7, Theorem 2.1], bounds the number of such germs in terms of an archimedean logarithmic energy and the p-adic radius; the numerical incompatibility of this bound with the count four is the contradiction.

What is new here is the level-five numerology and the exact treatment of its two analytic inputs. We derive the third-order equation explicitly from the Schwarzian derivative of the level-five uniformizer (Lemma 4.1) and prove by a monodromy argument that its inhomogeneous order is exactly three for every value of the constant term (Proposition 4.3); in [7, §5.2] the analogous exactness is quoted from Kontsevich–Zagier. We evaluate the archimedean energy of the level-five Hauptmodul in closed form by Jensen's formula (Lemma 6.1), and we certify the final inequality by rational arithmetic (Lemma 7.1). The margin is small, $3.9266\ldots$ against 4; Remark 7.2 discusses it. Without the row-wise optimization of the denominators the quotient would be $4.575\ldots$, and there would be no contradiction.

Outline of the paper

In Section 2 we introduce the level-five coordinate f, the modular form A, and the overconvergent Eisenstein germ $H(f) = A(q(f))\big(G(q(f)) + \zeta_5(3)/2\big)$. Under the contrary assumption $\zeta_5(3) \in \mathbf{Q}$, we have $H \in \mathbf{Q}[[f]]$. Section 3 states the adelic holonomy bound of Calegari–Dimitrov–Tang in the form in which it is applied and recalls its deduction from Bost's slopes inequality; the column-by-column denominator optimization is essential for the final constant. In Section 4 we derive an explicit third-order inhomogeneous differential equation for H and prove that its order is minimal. With $\delta = f\,d/df$, this gives the linear independence of $1, H, \delta H, \delta^2 H$ over $\mathbf{Q}(f)$. Section 5 proves the three denominator bounds of type D_n^3, where $D_n = \operatorname{lcm}(1, \ldots, n)$, and uses Buzzard's theorem to extend the three nonconstant germs to the strict disk $|f|_5 < 125$. Section 6 evaluates the archimedean boundary energy exactly by Jensen's formula on $X_0(5)$. Finally, Section 7 applies the holonomy bound to $1, H, \delta H, \delta^2 H$: the denominator rows give the optimized invariant $45/16$, and the local maps and energy calculation force the resulting quotient to be at least four. The rational numerical certificate proves that the same quotient is strictly smaller than four, yielding the required contradiction. Section 8 examines the level-seven analogue and the barriers to extending the method to general primes.

2 The level-five Eisenstein germ

Let $\tau \in \mathfrak{H}$, put $q = e^{2\pi i \tau}$, and write $\theta = q\,d/dq$. We use the conventions $\Delta = \eta^{24}$ and $\eta(\tau) = q^{1/24} \prod_{n \ge 1}(1 - q^n)$. Define

$$f = f_5(\tau) = \left(\frac{\Delta(5\tau)}{\Delta(\tau)}\right)^{1/4} = \frac{\eta(5\tau)^6}{\eta(\tau)^6} = q \prod_{n \ge 1} \left(\frac{1 - q^{5n}}{1 - q^n}\right)^6. \tag{1}$$

Here the fourth root is the one normalized by $f = q + O(q^2)$. We write $\mathfrak{f}(q)$ for the right-hand side of (1) regarded as an analytic function of q on $|q| < 1$. For the eta-quotient exponents $r_1 = -6$ and $r_5 = 6$, we have

$$\sum_{d|5} d r_d = 24, \qquad \sum_{d|5} \frac{5}{d} r_d = -24, \qquad \prod_{d|5} d^{r_d} = 5^6.$$

The eta-quotient criterion therefore shows that f is a weight-zero modular function on $\Gamma_0(5)$ with trivial character.

Let $w_5\tau = -1/(5\tau)$. The transformation formula $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$ gives

$$f(w_5\tau) = \frac{\eta(-1/\tau)^6}{\eta(-1/(5\tau))^6} = \frac{1}{125 f(\tau)}. \tag{2}$$

The product in (1) shows that f has a simple zero at ∞, while (2) shows that it has a simple pole at the cusp 0. The function η has no zeros on $\mathfrak{H}$, and these are the only two cusps of $X_0(5)$. Consequently

$$\operatorname{div}(f) = (\infty) - (0).$$

In particular, the map $f : X_0(5) \to \mathbf{P}^1$ has degree one; hence it is an isomorphism and f is a global coordinate on $X_0(5)$.

Since

$$\frac{1 - q^{5n}}{1 - q^n} = 1 + q^n + q^{2n} + q^{3n} + q^{4n},$$

we have $f \in q + q^2\mathbf{Z}[[q]]$. Recursive formal inversion of a series with linear coefficient one takes place over $\mathbf{Z}$, so $q(f) \in f + f^2\mathbf{Z}[[f]]$.

Define

$$A(\tau) = \frac{5E_2(5\tau) - E_2(\tau)}{4} = \frac{\theta f}{f} = 1 + 6\sum_{n\geq 1}\sum_{\substack{d|n \\ 5\nmid d}} d\, q^n, \tag{3}$$

$$G(q) = \sum_{n\geq 1}\left(\sum_{\substack{d|n \\ 5\nmid d}} d^{-3}\right) q^n, \tag{4}$$

$$\xi = \frac{\zeta_5(3)}{2}. \tag{5}$$

Indeed, $\theta\eta/\eta = E_2/24$ proves the first two expressions for A. Using $E_2(\tau) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$, its coefficient of q^n is

$$6\big(\sigma_1(n) - 5\sigma_1(n/5)\big) = 6\sum_{\substack{d|n \\ 5\nmid d}} d,$$

where $\sigma_1(n/5) = 0$ when $5 \nmid n$. This proves the last expression in (3). Equivalently, since f is a modular unit, its logarithmic derivative $A = \theta f/f$ is a holomorphic classical modular form of weight two on $\Gamma_0(5)$.

The p-adic Eisenstein family of Coleman and Mazur (see [8, §2.2] and [5, Theorem 2.3]) specializes at the character $x \mapsto x^{-2}$ to

$$E_{-2}^{*(5)} = \xi + G,$$

the q-expansion of an overconvergent modular form of weight -2 and level $\Gamma_0(5)$; compare [7, (5.4)]. With the normalization

$$U_5\left(\sum_{n\geq 0} a_n q^n\right) = \sum_{n\geq 0} a_{5n} q^n,$$

it has eigenvalue one: the divisors of $5n$ that are not divisible by five are precisely the divisors of n that are not divisible by five, and hence

$$\sum_{\substack{d|5n \\ 5\nmid d}} d^{-3} = \sum_{\substack{d|n \\ 5\nmid d}} d^{-3}.$$

The constant term ξ is also fixed by U_5. Thus this eigenform has slope zero.

The product

$$H(f) = A(q(f))\big(G(q(f)) + \xi\big) \tag{6}$$

is therefore the germ at $f = 0$ of a weight-zero overconvergent modular function. In the remainder of the proof we assume, for contradiction, that $\zeta_5(3) \in \mathbf{Q}$. Then $\xi \in \mathbf{Q}$; since $A, G \in \mathbf{Q}[[q]]$ and $q(f) \in \mathbf{Z}[[f]]$, equation (6) implies $H \in \mathbf{Q}[[f]]$.

3 The adelic holonomy bound

We isolate the precise slopes result that will be applied. It is the case in which the set S of approximation places is empty of the adelic holonomy bound [7, Theorem 2.6] of Calegari, Dimitrov and Tang, with the row-wise denominator invariant $\tau(\mathbf{b})$ of [7, Theorem 2.1]; the archimedean theory and its Bost–Charles metric are developed in [6]. Since [7, Theorem 2.6] is formulated for a nonempty S, and since the auxiliary denominator lattice must be optimized one column at a time, we reproduce the deduction of [7, §2.3], without approximation places, in the notation of this paper. Nothing in this section is new. Throughout, $|p|_p = p^{-1}$, and "holonomic" means that the series satisfies a nonzero linear differential equation over $\mathbf{Q}(x)$.

For $t \geq 0$, write

$$\Lambda(t) = \operatorname{lcm}(1, 2, \ldots, \lfloor t \rfloor), \qquad \Lambda(t) = 1 \quad (0 \leq t < 1).$$

Let $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$, equipped with normalized Haar probability measure μ.

Theorem 3.1 (adelic holonomy bound with row-wise denominators; [7, Theorem 2.6])**.** *Let $m, r \geq 1$, and let $F_1, \ldots, F_m \in \mathbf{Q}[[x]]$ be holonomic and linearly independent over $\mathbf{Q}(x)$. Suppose their coefficient denominators are capped by an $m \times r$ matrix $\mathbf{b} = (b_{i,j})$ whose j-th column has the form*

$$b_{1,j} = \cdots = b_{u_j,j} = 0, \qquad b_{u_j+1,j} = \cdots = b_{m,j} = b_j > 0,$$

where $0 \leq u_j \leq m$ and an empty string of equalities is omitted. More precisely, assume that there is an integer $C_0 \geq 1$, independent of i and n, such that

$$C_0 F_i(x) = \sum_{n\geq 0} \frac{a_{i,n} x^n}{\prod_{j=1}^{r} \Lambda(b_{i,j} n)}, \qquad a_{i,n} \in \mathbf{Z}. \tag{7}$$

For each place v of $\mathbf{Q}$, let φ_v be analytic on a neighborhood of the closed unit disk, defined over $\mathbf{Q}_v$ at a finite place and over $\mathbf{C}$ at infinity, with $\varphi_v(0) = 0$ and $\varphi_v'(0) \neq 0$. Suppose that $\varphi_p(z) = z$ for all but finitely many primes and that every $F_i \circ \varphi_v$ is meromorphic on the open unit disk. At a *finite place, disks are rigid-analytic disks after base change to $\mathbf{C}_p$, the completion of an algebraic closure of $\mathbf{Q}_p$; boundary suprema mean Gauss norms, equivalently suprema over $\mathbf{C}_p$-points. Put*

$$\mathcal{A} = \iint_{\mathbf{T}^2} \log|\varphi_\infty(z) - \varphi_\infty(w)|\, d\mu(z)\, d\mu(w) + \sum_p \sup_{|z|*p=1} \log|\varphi_p(z)|p, \tag{8}$$

$$\mathcal{L} = \log|\varphi\infty'(0)| + \sum_p \log|\varphi_p'(0)|p, \tag{9}$$

$$\tau(\mathbf{b}) = \sum j = 1^r b_j - \frac{1}{m^2} \sum *j = 1^r u_j^2 b_j. \tag{10}$$

All sums over finite primes are finite because $\varphi_p(z) = z$ almost everywhere. The double integral is finite: φ_∞ is nonconstant, and the zero set of $\varphi_\infty(z) - \varphi_\infty(w)$ produces only locally integrable logarithmic singularities on $\mathbf{T}^2$. If $\sigma_i = \sum_j b_{i,j}$ is the i-th row sum, then

$$\tau(\mathbf{b}) = \frac{1}{m^2}\sum_{i=1}^m (2i-1)\sigma_i, \qquad 0 \le \tau(\mathbf{b}) \le \sum_{j=1}^r b_j. \tag{11}$$

Indeed, the contribution of column j to the middle expression is $m^{-2} b_j \sum_{i=u_j+1}^m (2i-1) = b_j(1 - u_j^2/m^2)$. If $\mathcal{L} > \tau(\mathbf{b})$, then

$$m \le \frac{\mathcal{A}}{\mathcal{L} - \tau(\mathbf{b})}. \tag{12}$$

Proof. We follow [7, §2.3]. Replacing each F_i by $C_0 F_i$ does not affect any hypothesis or the conclusion. We may therefore take $C_0 = 1$ in (7); a fixed scaling would in any event contribute only $O(D)$ to the sum of evaluation heights below.

Fix real parameters $0 \le y_j \le m b_j$. For an integer $D \ge 1$, let $\mathbf{Z}[1/x] * \le D$ be the free module with basis $1, x^{-1}, \dots, x^{-D}$, and define the rank-$m(D+1)$ lattice

$$E_D = \bigoplus *i = 1^m \left(\prod_{j: i \le u_j} \Lambda(y_j D)^{-1} \right) e_i \mathbf{Z}[1/x]_{\le D}, \tag{13}$$

with evaluation map

$$\psi_D : E_D \otimes \mathbf{Q} \longrightarrow x^{-D}\mathbf{Q}[[x]], \qquad e_i P(x) \longmapsto P(x) F_i(x).$$

This map is injective: an element of its kernel would be a nontrivial $\mathbf{Q}(x)$-linear relation among the F_i.

We recall the metrics of the adelic slopes construction [7, §2.3]. At the archimedean place, the underlying rational blocks $e_i \mathbf{Q}[1/x] * \le D$ are mutually orthogonal and carry the L^2-metric induced by the Bost–Charles metric attached to $\varphi * \infty$ [6, §7.3]. At a finite prime p, put

$$s_p = \sup_{|z|*p=1} \log|\varphi_p(z)|p.$$

Write $\lambda_i(D) = \prod j : i \le u_j \Lambda(y_j D)$. Give the lattice basis vector $\lambda_i(D)^{-1} e_i x^{-k}$ the diagonal norm $e^{-k s_p}$; equivalently,

$$\left\| \sum *i, k c_{i,k} e_i x^{-k} \right\| * p = \max *i, k |\lambda_i(D) c_{i,k}| * p e^{-k s_p}.$$

For all but finitely many primes this is the standard lattice norm. The arithmetic Hilbert–Samuel formula and the Bost–Charles self-intersection calculation [3, Theorem 5.4.1 and Proposition 5.4.2] give

$$\widehat{\deg}\,\overline{E}_D = \left(\frac{m}{2}\mathcal{A} + \sum *j = 1^r u_j y_j\right) D^2 + o(D^2). \tag{14}$$

Indeed, if $\mathcal{A}_\infty$ denotes the double integral in equation (8), the archimedean contribution of one block is $\mathcal{A}_\infty D^2/2 + o(D^2)$. At p, the determinant of the diagonal metric contributes $s_p \sum_{k=0}^{D} k = s_p D^2/2 + O(D)$ per block. Finally, $\Lambda(y_j D)^{-1}$ occurs in $u_j(D+1)$ lattice basis vectors, and $\log \Lambda(y_j D) = y_j D + o(D)$ by the prime number theorem. These three contributions give equation (14).

For $n \geq 0$, filter the target and the source by

$$\mathcal{F}_D^{(n)} = x^{n-D}\mathbf{Q}[[x]], \qquad E_D^{(n)} = E_D \cap \psi_D^{-1}(\mathcal{F}_D^{(n)}).$$

Each graded piece $E_D^{(n)}/E_D^{(n+1)}$ has rank zero or one. Let

$$\mathcal{V}_D = \{n \geq 0 : \operatorname{rank}(E_D^{(n)}/E_D^{(n+1)}) = 1\}.$$

The filtration is separated and exhaustive, so $|\mathcal{V}_D| = \operatorname{rank} E_D = m(D+1)$. For $n \in \mathcal{V}_D$, evaluation induces an injection

$$\psi_D^{(n)} : E_D^{(n)}/E_D^{(n+1)} \longrightarrow \mathcal{F}_D^{(n)}/\mathcal{F}_D^{(n+1)}.$$

Write $h_v(\psi_D^{(n)})$ for its local height with respect to the metrics just specified and the norm-one generator x^{n-D} of the target.

The Chudnovsky–Osgood functional bad-approximability theorem for an independent holonomic system [6, Theorem 3.2.13] implies that, for every $\epsilon > 0$, there is a constant C_ϵ such that

$$\mathcal{V}_D \subset \{0, 1, \ldots, \lfloor (m+\epsilon)(D+1) + C_\epsilon \rfloor\}. \tag{15}$$

The Bost–Charles subharmonic estimate at infinity and its nonarchimedean maximum-principle analogue at the finite places [7, Lemma 2.9] give, for $n \in \mathcal{V}_D$,

$$\sum_v h_v(\psi_D^{(n)}) \leq -n\mathcal{L} + D\mathcal{A} + \sum_{j=1}^{r} b_j \max\left\{n, \frac{y_j}{b_j} D\right\} + o(n+D). \tag{16}$$

The hypothesis of meromorphy on the open disk is sufficient here [6, Lemma 7.4.1]. Restrict to a closed subdisk of radius $\rho < 1$, rescale it to the unit disk, clear the finitely many poles there by a common factor equal to one at the origin, and apply the closed-disk estimate. The clearing factor contributes only $O(1)$ to each local height. One first lets $D \to \infty$ and then $\rho \uparrow 1$; this is the standard limiting argument of [7, §2.3].

We spell out the denominator term, which is the only modification of the standard local estimates. Write $P_i = \sum_{k=0}^{D} c_{i,k} x^{-k}$. A contribution to the coefficient of x^{n-D} in $P_i F_i$ uses the coefficient of F_i with index $n - D + k$; whenever it occurs, this index lies between 0 and n. Thus, in column j, a row $i > u_j$ has coefficient denominator dividing $\Lambda(b_j n)$, whereas a row $i \leq u_j$ has no denominator from F_i in that column but has the lattice denominator $\Lambda(y_j D)$. The least common multiple over all rows divides $\Lambda\big(\max\{b_j n, y_j D\}\big)$, because the smaller of two initial-segment least common multiples divides the larger. Summing its valuations over the finite primes and using the prime number theorem contributes $b_j \max\{n, (y_j/b_j)D\} + o(n+D)$. The analytic parts of the local estimates supply $-n\mathcal{L} + D\mathcal{A}$; see [7, Lemma 2.9] and [6, (7.3.5) and (7.4.3)]. More precisely, use the analytic estimate at the finitely many primes with nonidentity maps and the coefficient estimate at the remaining primes. At each fixed exceptional prime, the valuations of $\Lambda(b_j n)$ and $\Lambda(y_j D)$ are $O(\log(n+D+2))$. Consequently, inserting these primes into the denominator sum changes only the error term and justifies combining the full prime-number-theorem sum with the analytic contributions. For fixed ϵ and fixed $y_1, \dots, y_r$, the error in equation (16) is uniform for the range of n in (15).

We now sum the local estimates. Since $\mathcal{L} > \tau(\mathbf{b}) \geq 0$, the coefficient of n in the first term is negative. The $m(D+1)$ jumps are distinct nonnegative integers, and therefore

$$\sum_{n \in \mathcal{V}_D} n \geq \frac{m(D+1)\big(m(D+1)-1\big)}{2} = \frac{m^2}{2} D^2 + o(D^2). \tag{17}$$

For the increasing function $g_j(t) = b_j \max\{t, y_j/b_j\}$, the largest sum over $m(D+1)$ distinct integers in the interval from (15) uses its $m(D+1)$ largest integers. After division by D^2, this is an upper Riemann sum on $[\epsilon, m+\epsilon]$, up to $o(1)$. Letting $D \to \infty$ and then $\epsilon \downarrow 0$ gives

$$\limsup_{D \to \infty} \frac{1}{D^2} \sum_{n \in \mathcal{V}_D} b_j \max\left\{n, \frac{y_j}{b_j} D\right\} \leq b_j \int_0^m \max\left\{t, \frac{y_j}{b_j}\right\} dt. \tag{18}$$

The uniform error in (16) sums to $o(D^2)$. In the present rank-one graded setting, Bost's slopes inequality is

$$\widehat{\deg}\, \overline{E}_D \leq \sum_{n \in \mathcal{V}_D} \sum_v h_v(\psi_D^{(n)});$$

see [2, §§4.1–4.2]. Combining this inequality with equations (17) and (18) yields

$$\widehat{\deg}\, \overline{E}_D \leq \left(-\frac{m^2}{2}\mathcal{L} + m\mathcal{A} + \sum_{j=1}^r b_j \int_0^m \max\left\{t, \frac{y_j}{b_j}\right\} dt \right) D^2 + o(D^2). \tag{19}$$

Here the term $m\mathcal{A}D^2$ is simply $|\mathcal{V}_D| D \mathcal{A} + o(D^2)$. Since $0 \leq y_j/b_j \leq m$,

$$b_j \int_0^m \max\left\{t, \frac{y_j}{b_j}\right\} dt = \frac{1}{2}\left(m^2 b_j + \frac{y_j^2}{b_j}\right),$$

comparison of (14) and (19) yields

$$m^2\left(\mathcal{L} - \sum_j b_j\right) \leq m\mathcal{A} + \sum_j \left(\frac{y_j^2}{b_j} - 2u_j y_j\right). \tag{20}$$

The right side is minimized at $y_j = u_j b_j$, which lies in the allowed range $[0, m b_j]$. Substitution in (20) gives

$$m^2\left(\mathcal{L} - \sum_j b_j\right) \leq m\mathcal{A} - \sum_j u_j^2 b_j,$$

or equivalently $m(\mathcal{L} - \tau(\mathbf{b})) \leq \mathcal{A}$. The assumed strict positivity of $\mathcal{L} - \tau(\mathbf{b})$ now permits division and proves equation (12). $\square$

Remark 3.2. The crucial feature is the subtraction $m^{-2}\sum_j u_j^2 b_j$ in $\tau(\mathbf{b})$. It comes from minimizing the quadratic in (20) before the columns are combined. The same invariant governs the proof of the irrationality of $\zeta_2(5)$ in [7, §5.2], where $m = 6$, the capping matrix has rows $(0), (5), (5), (5), (5), (5)$, and $\tau(\mathbf{b}) = 175/36$.

4 The differential equation and its exact order

Put $\delta = f\,d/df$ and

$$P = 1 + 22f + 125f^2, \qquad Q = 1 + 23f + 125f^2, \qquad T = 1 + 24f + 125f^2,$$
$$N = 1 + 250f + 3125f^2, \qquad M = 1 + 10f + 5f^2.$$

We first establish the modular identities used below. At the cusp ∞, the functions $j(\tau)$ and $j(5\tau)$ have pole orders one and five, respectively. At the cusp 0, their pole orders are five and one, respectively: indeed, $j(w_5\tau) = j(5\tau)$ and $j(5w_5\tau) = j(\tau)$. Since f has divisor $(\infty) - (0)$, the functions $j(\tau)f$ and $j(5\tau)f^5$ are regular away from 0 and have there a pole of order at most six. They are therefore polynomials in f of degree at most six. Exact formal expansion through degree six at $f = 0$, obtained from (1), $j = E_4^3/\Delta$, and the recursively determined $q(f)$, gives

$$j(\tau)f = N^3 + O(f^7), \qquad j(5\tau)f^5 = M^3 + O(f^7).$$

The degree bound makes this finite comparison conclusive. Thus the Hauptmodul identities, in the normalization (1), are

$$j(\tau) = \frac{N^3}{f}, \qquad j(5\tau) = \frac{M^3}{f^5}. \tag{21}$$

For a direct derivation of the weight-four identities, put

$$R_1 = 15625f^2 + 500f - 1, \qquad R_2 = f^2 - 4f - 1.$$

Logarithmic differentiation of (21) gives

$$\delta \log j(\tau) = \frac{R_1}{N}, \qquad \frac{1}{5}\delta \log j(5\tau) = \frac{R_2}{M}.$$

Using $\theta = A\delta$ and the identities

$$\frac{\theta j(\tau)}{j(\tau)} = -\frac{E_6(\tau)}{E_4(\tau)}, \qquad \frac{\theta j(5\tau)}{j(5\tau)} = -5\frac{E_6(5\tau)}{E_4(5\tau)}, \qquad \frac{E_6^2}{E_4^2} = E_4\left(1 - \frac{1728}{j}\right),$$

together with the exact polynomial factorizations

$$\begin{aligned} N^3 - 1728f &= PR_1^2, \\ M^3 - 1728f^5 &= PR_2^2, \end{aligned} \tag{22}$$

one obtains

$$\frac{E_4(\tau)}{A^2} = \frac{N}{P}, \qquad \frac{E_4(5\tau)}{A^2} = \frac{M}{P}, \qquad \frac{E_4(\tau) - E_4(5\tau)}{240} = A^2\frac{f(1+13f)}{P}. \tag{23}$$

For example, the first identity is the exact simplification

$$\frac{E_4(\tau)}{A^2} = \frac{(R_1/N)^2}{1 - 1728f/N^3} = \frac{N}{P};$$

the second is identical with R_2, M. The third follows from the first two and $N - M = 240f(1+13f)$. All comparisons above take place in $\mathbf{Z}[[q]]$ or $\mathbf{Q}(f)$; no numerical recognition is involved.

Define

$$\mathcal{L}_3 = P^3\delta^3 - 12fPQ\,\delta + 6f(125f^2 - 1)T. \tag{24}$$

Lemma 4.1 (exact modular equation)**.** *On an open set where $A \neq 0$, every holomorphic germ $g(\tau)$ satisfies*

$$\mathcal{L}_3(Ag) = \frac{P^3}{A^2}\theta^3 g. \tag{25}$$

Consequently,

$$\mathcal{L}_3 H = f(1+13f)P^2. \tag{26}$$

Proof. Put $u = \delta A/A$ and $s = \delta\tau = (2\pi i A)^{-1}$, so $\delta s/s = -u$. With $z = 1728/j = 1728f/N^3$, the inverse uniformizing map is the ratio of two solutions, up to a fractional-linear transformation, of the hypergeometric equation

$$z(1-z)y'' + \left(1 - \frac{3}{2}z\right)y' - \frac{5}{144}y = 0.$$

The standard Schwarzian formula for a ratio of solutions therefore gives

$$\{\tau, z\} = \frac{32z^2 - 41z + 36}{72z^2(1-z)^2}.$$

Here $\{h, x\} = h'''/h' - \frac{3}{2}(h''/h')^2$ denotes the Schwarzian derivative, with derivatives taken with respect to x. The Schwarzian chain rule gives $\{\tau, \log f\} = \{\tau, z\}(\delta z)^2 + \{z, \log f\}$, where $\delta = d/d\log f$. Substitution of $z = 1728f/N^3$ and exact simplification yield the first identity below. On the other hand, $\delta\tau = s$ and $\delta s = -us$ give $\{\tau, \log f\} = -\delta u - u^2/2$, so

$$\{\tau, \log f\} = -\frac{6fQ}{P^2}, \qquad 2\delta u + u^2 = \frac{12fQ}{P^2}. \tag{27}$$

Differentiating and using

$$\delta\left(\frac{fQ}{P^2}\right) = \frac{f(1-125f^2)T}{P^3}$$

gives $\delta^2 u + u\,\delta u = 6f(1-125f^2)T/P^3$. Also,

$$\frac{\delta^3 A}{A} = \delta^2 u + 3u\,\delta u + u^3 = (\delta^2 u + u\,\delta u) + u(2\delta u + u^2). \tag{28}$$

Writing primes for $d/d\tau$, three applications of the product rule give $\delta^3(Ag) = (\delta^3 A)g + As(2\delta u + u^2)g'(\tau) + As^3 g'''(\tau)$; the g''-coefficient cancels because $\delta s = -us$. Substitution in (24), using (27) and (28), cancels the g- and g'-terms. Finally, $As^3 g''' = A^{-2}\theta^3 g$, so the remaining term is $P^3 A^{-2}\theta^3 g$, proving (25).

Finally, coefficientwise differentiation of (4) gives

$$\theta^3 G = \frac{E_4(\tau) - E_4(5\tau)}{240}.$$

Indeed, the coefficient of q^n on both sides is $\sum_{d|n,\, 5\nmid d}(n/d)^3$. Equations (23) and (25), first with $g = 1$ and then with $g = G$, prove (26). □

Definition 4.2. The *inhomogeneous differential order* of a germ Y over $\mathbf{C}(f)$ is the least integer $r \geq 0$ for which

$$\delta^r Y + a_{r-1}\delta^{r-1}Y + \cdots + a_0 Y = b, \qquad a_i, b \in \mathbf{C}(f).$$

For $r = 0$, this means simply $Y = b$. If no such r exists, the order is infinite. Equivalently, when the order is finite, it is one less than the dimension of the $\mathbf{C}(f)$-span of $1, Y, \delta Y, \ldots$: the first dependence in this ordered list is precisely a normalized inhomogeneous equation of minimal order.

Proposition 4.3 (exact inhomogeneous order)**.** *For every $c \in \mathbf{C}$, the germ $Y_c = A(G+c)$ has exact inhomogeneous order three over $\mathbf{C}(f)$; equivalently, $1, Y_c, \delta Y_c, \delta^2 Y_c$ are linearly independent over $\mathbf{C}(f)$, and all four germs are holonomic. In particular*

$$1, H, \delta H, \delta^2 H$$

are linearly independent over $\mathbf{Q}(f)$ and holonomic. No assumption on ξ is needed for this.

Proof. Fix $c \in \mathbf{C}$ and put $Y_c = A(G+c)$. Since $\mathcal{L}_3 A = 0$, Lemma 4.1 gives

$$\mathcal{L}_3 Y_c = f(1+13f)P^2, \tag{29}$$

so the inhomogeneous order is at most three.

Equation (25) shows that the local homogeneous solution space V of $\mathcal{L}_3$ is

$$A\langle 1, \tau, \tau^2 \rangle.$$

Under modular continuation this is the symmetric square of the standard two-dimensional representation. Its monodromy is irreducible: $\Gamma_0(5)$ has finite index in $\mathrm{SL}_2(\mathbf{Z})$, hence is Zariski dense in SL_2, and Sym^2 of the standard representation is irreducible.

We next prove that equation (29) has no rational particular solution. Thus there is no $R \in \mathbf{C}(f)$ satisfying

$$\mathcal{L}_3 R = f(1+13f)P^2. \tag{30}$$

The discriminant of P is -16, so its two roots are distinct. At either root f_0, substitution of $(f-f_0)^\lambda$ into the lowest-order terms uses

$$Q(f_0) = f_0, \qquad T(f_0) = 2f_0, \qquad (f_0 P'(f_0))^2 = -16 f_0^2, \qquad 125 f_0^2 - 1 = f_0 P'(f_0),$$

and the indicial polynomial, normalized by division by $(f_0 P'(f_0))^3$, is

$$\Phi_{f_0}(\lambda) = \lambda(\lambda-1)(\lambda-2) + \frac{3}{4}\lambda - \frac{3}{4} = \left(\lambda - \frac{1}{2}\right)(\lambda - 1)\left(\lambda - \frac{3}{2}\right). \tag{31}$$

The right side of (30) vanishes to order two at f_0. Since no negative integer is a root of (31), a rational solution cannot have a pole at either root of P.

At a finite point $f_1 \neq 0$ with $P(f_1) \neq 0$, the coefficient of the most singular term of a putative principal part $c_0(f-f_1)^{-k}$, $k > 0$, is $P(f_1)^3 f_1^3 c_0(-k)(-k-1)(-k-2)(f-f_1)^{-k-3}$, which is nonzero. At $f = 0$, a term $c_0 f^{-k}$ contributes $c_0(-k)^3 f^{-k}$ through $P^3\delta^3$, while the other terms have strictly larger f-order. Thus a rational solution has no finite poles and must be a polynomial.

If its degree is $d > 0$, the unique term of degree $d+6$ is the leading term of $P^3\delta^3 R$, with nonzero coefficient $125^3 d^3$ times the leading coefficient of R. The other terms have degree at most $d+5$, whereas the right side of (30) has degree six. This is impossible. If R is constant, the left side has degree at most five, again contradicting the degree-six right side. This proves the claim.

We now record the regular-singular property needed for the monodromy argument. The only singular points of the homogeneous equation on $\mathbf{P}^1_f$ are $0, \infty$, and the two roots of P. At a root of P, passage to the usual monic equation gives a holomorphic coefficient of the second derivative, a pole of order at most two in the coefficient of the first derivative, and a pole of order at most three in the zeroth-order coefficient. Hence these two points are regular singular; regular singularity at 0 is immediate from the Euler form. At infinity, divide by P^3 and put $t = 1/f$. The coefficients of δ and of the zeroth-order term are, respectively,

$$-\frac{12}{125}t + O(t^2), \qquad \frac{6}{125}t + O(t^2),$$

while the normalized inhomogeneous term is

$$\frac{f(1+13f)}{P} = \frac{13}{125} + O(t).$$

Thus infinity is regular singular as well. The normalized inhomogeneous term is meromorphic at the other three singularities, so the usual Frobenius theorem applies to local branches of Y_c.

Let X be the complement of these four points, fix a branch of Y_c on a simply connected open subset of X, and let ρ denote the monodromy representation on V. For a loop γ, the branch difference $\kappa_\gamma = Y_c^\gamma - Y_c$ lies in V. These differences cannot all vanish. Otherwise Y_c would be single-valued on X. Since the homogeneous equation is Fuchsian at each of the four punctures and the inhomogeneous term is a polynomial, every local solution of (29) has moderate growth near each puncture: the homogeneous solutions by Fuchs's theorem, and a particular solution by variation of parameters. A single-valued function of moderate growth near an isolated singularity is meromorphic there. Thus Y_c would extend meromorphically to $\mathbf{P}^1$, hence would be rational. This contradicts the preceding exclusion of a rational particular solution.

Consequently, the span $W = \langle \kappa_\gamma \rangle$ is nonzero. The cocycle identity $\kappa_{\alpha\beta} = \kappa_\alpha + \rho(\alpha)\kappa_\beta$ shows that W is monodromy-stable. Irreducibility of V therefore gives $W = V$, so the branch differences span the full three-dimensional homogeneous solution space.

Any inhomogeneous equation of order at most two would force all branch differences into a homogeneous solution space of dimension at most two. Here the continuation loops may also avoid the finitely many additional poles of the proposed equation; this does not change the monodromy generated by the original punctures. Together with the upper bound from (29), this proves that Y_c has exact inhomogeneous order three for every $c \in \mathbf{C}$. By the remark following the definition, $1, Y_c, \delta Y_c, \delta^2 Y_c$ are linearly independent over $\mathbf{C}(f)$.

Finally, put $\mathcal{J} = f(1+13f)P^2$. The first-order operator $\mathcal{J}\delta - (\delta\mathcal{J})$ annihilates $\mathcal{J}$, so its composition with $\mathcal{L}_3$ is a homogeneous operator annihilating Y_c. Thus Y_c, its derivatives, and 1 are holonomic. For the assertion about H, choose a field embedding $\iota : \mathbf{Q}(\xi) \hookrightarrow \mathbf{C}$ and set $c = \iota(\xi)$. Such an embedding exists whether ξ is algebraic or transcendental over $\mathbf{Q}$. Applying ι coefficientwise sends H to Y_c and commutes with δ. Any dependence over $\mathbf{Q}(f)$ among $1, H, \delta H, \delta^2 H$ would therefore give the forbidden dependence among $1, Y_c, \delta Y_c, \delta^2 Y_c$. The homogeneous operator just constructed has coefficients in $\mathbf{Q}(f)$, is independent of c, and also annihilates H as a formal series. This proves both assertions without assuming that ξ is rational. $\square$

5 Denominators and local continuation

Set $D_0 = 1$, and for $n \geq 1$ set $D_n = \operatorname{lcm}(1, 2, \ldots, n)$.

Lemma 5.1 (denominator rows)**.** *Write*

$$A(q(f)) = \sum_{n\geq 0} \beta_n f^n, \qquad A(q(f))G(q(f)) = \sum_{n\geq 0} \alpha_n f^n.$$

Then

$$\beta_n \in \mathbf{Z}, \qquad D_n^3 \alpha_n \in \mathbf{Z}.$$

Under the rationality assumption, write $\xi = a_0/b_0$, where $a_0, b_0 \in \mathbf{Z}$ and $b_0 > 0$. Then, for $r = 0, 1, 2$ and $n \geq 0$,

$$b_0 D_n^3 [f^n] \delta^r H \in \mathbf{Z}.$$

Here $[f^n]$ denotes coefficient extraction.

Proof. By (3) and the integrality of $q(f)$,

$$B(f) := A(q(f)) \in \mathbf{Z}[[f]],$$

which proves the assertion for β_n. Write $G(q) = \sum_{m\geq 1} g_m q^m$. Every divisor d occurring in g_m satisfies $d \leq m$, hence $d \mid D_m$, so $D_m^3 g_m \in \mathbf{Z}$. Since $q(f)^m \in f^m\mathbf{Z}[[f]]$, we have

$$[f^n]G(q(f)) = \sum_{m=1}^{n} g_m [f^n] q(f)^m.$$

The bracketed coefficients are integers, and $D_m \mid D_n$ for $m \leq n$. It follows that $D_n^3[f^n]G(q(f)) \in \mathbf{Z}$.

Now $\alpha_n = [f^n]\big(B(f)G(q(f))\big)$. In its Cauchy product, the coefficient from degree s of $G(q(f))$ is cleared by D_s^3, which divides D_n^3 whenever $s \leq n$; the coefficient from B is integral. Hence $D_n^3\alpha_n \in \mathbf{Z}$. Finally, if $H = \sum_{n\geq 0} h_n f^n$, then

$$h_n = \alpha_n + \frac{a_0}{b_0}\beta_n, \qquad [f^n]\delta^r H = n^r h_n \quad (r = 1, 2),$$

while the coefficient is h_n for $r = 0$. Thus $b_0 D_n^3 [f^n]\delta^r H$ is integral in all three cases, as claimed. □

Lemma 5.2 (the 5-adic continuation disk)**.** *The functions $H, \delta H, \delta^2 H$ are analytic on $|f|_5 < 125$.*

Proof. Normalize v_5 by $v_5(5) = 1$. We first identify the relevant wide open. From (21), if $t = v_5(f)$, the three terms of $N = 1 + 250f + 3125f^2$ have valuations

$$0, \qquad 3 + t, \qquad 5 + 2t.$$

For $t \geq 0$, the first is the unique minimum and $v_5(N) = 0$. For $-3 < t < -5/2$, the last is the unique minimum and $v_5(N) = 5 + 2t$. For $-5/2 < t < 0$, the first is again the unique minimum; at $t = -5/2$, possible cancellation can only increase $v_5(N)$. Finally, for $t \leq -3$, the last term is the unique minimum. Since $j = N^3/f$, this proves

$$v_5(j) = -t \quad (t \geq 0), \qquad v_5(j) > 0 \quad (-3 < t < 0), \qquad v_5(j) = 15 + 5t \quad (t \leq -3), \tag{32}$$

where in the middle range only the strict inequality is asserted.

The unique supersingular j-invariant in characteristic five is 0: indeed, the Hasse invariant is E_4 mod 5, and $j = E_4^3/\Delta$ on the smooth locus. Consequently, $v_5(j) > 0$ is precisely the tube of the supersingular point, whereas the two ranges with $v_5(j) \leq 0$ give the two ordinary components and their cusp neighborhoods. Thus the ordinary component containing ∞, the ordinary component containing 0, and the intervening supersingular annulus are, respectively,

$$|f|_5 \leq 1, \qquad |f|_5 \geq 125, \qquad 1 < |f|_5 < 125. \tag{33}$$

The Fricke relation (2) interchanges the first two regions and provides an independent check of their endpoints.

By the U_5-calculation in Section 2, the overconvergent form $E_{-2}^{*(5)}$ is a U_5-eigenform with eigenvalue one. We apply Buzzard's analytic-continuation theorem [4, Theorem 5.2(2)]. It is stated for an overconvergent eigenform of arbitrary integral weight k and tame level $N \geq 5$ prime to p, with nonzero U_p-eigenvalue. On $X_1(N; p)$, the modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$, it extends the form to a section of $\omega^{\otimes k}$ over the wide open $\mathcal{W}_0(p)$ of points whose reduction lies on the irreducible component of the special fibre containing the cusp ∞ [4, §4]. This region is the complement of the ordinary component containing the cusp 0, and it contains the entire supersingular locus. By the first remark following [4, Theorem 5.2], the result holds for any sufficiently fine tame level structure. Negative weights are allowed.

Since our form has tame level one, we choose an auxiliary integer $N \geq 5$ prime to 5 and pull back $E_{-2}^{*(5)}$ and the classical form A to $X_1(N;5)$. The pullback of $E_{-2}^{*(5)}$ is an overconvergent U_5-eigenform of weight -2 with eigenvalue one, and [4, Theorem 5.2(2)] extends it to a section of $\omega^{\otimes -2}$ over $\mathcal{W}_0(5) \subset X_1(N;5)^{\mathrm{an}}$. The pullback of A is a global section of $\omega^{\otimes 2}$. Their product is a rigid-analytic function $\widetilde{H}$ on $\mathcal{W}_0(5)$ whose q-expansion at the cusp ∞ is $A(q)\big(G(q)+\xi\big)$, the q-expansion of H. Let $\pi : X_1(N;5)^{\mathrm{an}} \to X_0(5)^{\mathrm{an}}$ be the finite forgetful morphism, of degree d. It is flat, since it is a finite morphism between smooth curves. The description of $\mathcal{W}_0(5)$ is independent of the auxiliary tame level: forgetting that level preserves the ordinary component containing ∞ and the supersingular locus. Indeed, on the ordinary locus membership in that component is determined by the canonical order-five subgroup, independently of the prime-to-five level structure [4, §4]. Thus $\mathcal{W}_0(5) = \pi^{-1}(\mathcal{D})$, where $\mathcal{D} \subset X_0(5)^{\mathrm{an}}$ is the union of the ordinary component containing ∞ and the supersingular annulus. By (33), this is the strict disk

$$\mathcal{D} = \{\, |f| * 5 < 125 \,\}. \tag{34}$$

The finite-flat trace defines a rigid-analytic function

$$H * \mathrm{ext} = \frac{1}{d}\,\mathrm{Tr} * \pi(\widetilde{H}) \quad \text{on } \mathcal{D};$$

see the trace construction in [4, §5]. On the original overconvergent neighborhood $\mathcal{U}$ downstairs, the construction gives $\widetilde{H} = \pi^* H$ on $\pi^{-1}(\mathcal{U})$. Since $\mathrm{Tr} * \pi(\pi^* H) = dH$, we have $H_{\mathrm{ext}} = H$ on $\mathcal{U}$. Its Taylor series at $f = 0$ is therefore the original germ H. Analytic functions on an open disk are represented by their convergent Taylor series, so H is analytic on $|f|_5 < 125$. Since f is a coordinate, d/df, and hence $\delta = f\,d/df$, preserves analyticity on the open disk. Thus δH and $\delta^2 H$ are analytic there as well. No convergence or continuation on the boundary $|f|_5 = 125$ is asserted. ☐

Still under the standing rationality assumption, at a prime $\ell \neq 5$, Lemma 5.1 gives convergence of $H, \delta H, \delta^2 H$ on $|f| * \ell < 1$. Indeed, the fixed factor b_0 does not affect the radius, and, for $n \geq 1$,

$$v * \ell(D_n) = \max\{e : \ell^e \leq n\} = \lfloor \log_\ell n \rfloor = O(\log n).$$

Thus the n-th coefficient of any of the three germs has ℓ-adic valuation at least $-3\lfloor \log_\ell n\rfloor - v_\ell(b_0)$, which is $o(n)$.

At the complex place, the coefficients of $A(q)$ have polynomial growth, while the coefficient of q^n in $G(q)$ is at most $\sum_{d\geq 1} d^{-3} = \zeta(3)$. Hence both series are holomorphic for $|q| < 1$, and $H(\mathfrak{f}(q)) = A(q)(G(q)+\xi)$ is holomorphic there. Since $\delta = \theta/A$ after pullback to the q-disk, the pullbacks of δH and $\delta^2 H$ are meromorphic on $|q| < 1$, with poles at most at the zeros of A. These zeros are the critical points of f on $\mathfrak{H}$, that is, the $\Gamma_0(5)$-orbits of the two elliptic points $\tau_\pm = (\pm 2 + i)/5$ of $X_0(5)$, whose stabilizers are generated, modulo $\{\pm I\}$, by $\begin{pmatrix} \pm 2 & -1 \\ 5 & \mp 2 \end{pmatrix}$. For $\gamma \in \Gamma_0(5)$ with lower row (c,d) we have $\mathrm{Im}(\gamma\tau_\pm) = \frac{1}{5}|c\tau_\pm + d|^{-2}$, while $|q| \leq 1/2$ means $\mathrm{Im}\,\tau \geq \log 2/(2\pi)$, that is, $|c\tau_\pm + d|^2 \leq 2\pi/(5\log 2) = 1.813\ldots$. Since $|c\tau_\pm + d|^2 \geq (c/5)^2 \geq 4$ for $|c| \geq 10$, and $|c\tau_\pm + d|^2 = (d\pm 2)^2 + 1$ for $c = 5$, this forces $c = 0$, or $c = 5$ and $d = \mp 2$, after changing the sign of γ if needed. The resulting matrices are integer translations, possibly composed with the indicated stabilizer, so they give the same two q-values. Consequently the only zeros of A in the closed disk $|q| \leq 1/2$ are the two points $q_\pm = e^{2\pi i \tau_\pm}$, of modulus $e^{-2\pi/5} = 0.284\ldots$. Thus the pullbacks of δH and $\delta^2 H$ by $q \mapsto \mathfrak{f}(q)$ are meromorphic on a neighborhood of $|q| \le 1/2$ with poles at most at $q_\pm$. The meromorphy hypothesis of Theorem 3.1 is thus used in earnest at the complex place. At the finite places, only meromorphy on the open unit disk is needed or asserted for these pullbacks.

6 Exact complex logarithmic energy

Recall that

$$\mathfrak{f}(q) = q \prod_{n \ge 1} \left(\frac{1 - q^{5n}}{1 - q^n} \right)^6$$

denotes the analytic q-expansion in (1). Take

$$\varphi_\infty(z) = \mathfrak{f}(z/2), \qquad \varphi_5(z) = 5^{-3} z, \qquad \varphi_\ell(z) = z \quad (\ell \ne 5). \tag{35}$$

All these maps vanish at the origin and have nonzero derivative there. The first is analytic for $|z| < 2$, because $\mathfrak{f}(q)$ is analytic for $|q| < 1$. At five,

$$|5^{-3} z|_5 = 125|z| * 5 < 125 \qquad (|z|5 < 1),$$

so Lemma 5.2 applies. At every other finite place the identity map lands in the convergence disk established at the end of Section 5. That section also verifies the complex meromorphy required in Theorem 3.1: the pullbacks of the four germs by $\varphi\infty$ are meromorphic on a neighborhood of the closed unit disk, with poles at most at the two points $2q * \pm$, of modulus $2e^{-2\pi/5} = 0.569\ldots$.

Let

$$I = \iint_{\mathbf{T}^2} \log |\mathfrak{f}(z/2) - \mathfrak{f}(w/2)| \, d\mu(z) \, d\mu(w).$$

Lemma 6.1 (Modular Jensen energy)**.** *Put $\kappa = 5 \log 2/(2\pi)$ and $\nu = \sqrt{1 - \kappa^2}$. Then $0 < \kappa < 1$ and*

$$I = -\log 2 + \frac{16\pi}{25} \arccos \kappa - \frac{8 \log 2}{5} \, \nu. \tag{36}$$

Proof. Let $r = 1/2$, $y_0 = \log 2/(2\pi)$, and $q_u = e^{2\pi i(u + iy_0)} = re^{2\pi i u}$. We first justify the use of Fubini. The function $(z, w) \longmapsto \mathfrak{f}(rz) - \mathfrak{f}(rw)$ is holomorphic on a neighborhood of the compact torus $\mathbf{T}^2$ and is not identically zero. Its pullback under $(u, v) \mapsto (e^{2\pi i v}, e^{2\pi i u})$ is therefore a nonzero real-analytic function on a neighborhood of $[0, 1] \times [0, 1]$. The logarithm of the absolute value of a nonzero real-analytic function is locally integrable (this follows, for example, from Weierstrass preparation and one-variable Fubini). Compactness consequently gives $(u, v) \longmapsto \log \left|\mathfrak{f}(re^{2\pi i v}) - \mathfrak{f}(re^{2\pi i u})\right|$ in $L^1([0, 1] \times [0, 1])$, so Fubini's theorem applies.

For every u, the function $z \mapsto \mathfrak{f}(rz) - \mathfrak{f}(q_u)$ is holomorphic on a neighborhood of the closed unit disk, and its value at the origin is $-\mathfrak{f}(q_u) \ne 0$. The closed-circle form of Jensen's formula therefore gives

$$\int_0^1 \log |\mathfrak{f}(re^{2\pi i v}) - \mathfrak{f}(q_u)| \, dv = \log |\mathfrak{f}(q_u)| + \sum_{|z_0| < 1} \log \frac{1}{|z_0|}. \tag{37}$$

Here the zeros are counted with multiplicity. Zeros on the unit circle give integrable logarithmic singularities but no term in the sum. Because $\mathfrak{f}(q)/q$ is holomorphic and nonzero for $|q| < 1$, and has value one at $q = 0$, the mean of $\log |\mathfrak{f}(q)/q|$ on $|q| = r$ is zero. Hence

$$\int_0^1 \log |\mathfrak{f}(q_u)| \, du = \log r = -\log 2.$$

This is the contribution of the first term on the right of (37) after integration in u.

Since f is a degree-one coordinate on $X_0(5)$, we have $f(\tau') = f(\tau)$ for $\tau', \tau \in \mathfrak{H}$ exactly when $\tau' = \gamma\tau$ for some $\gamma \in \Gamma_0(5)$. For nonelliptic τ, the distinct q-coordinate preimages are therefore indexed by $\Gamma_\infty\backslash\Gamma_0(5)$. Representatives of these cosets, where

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbf{Z} \right\},$$

are parametrized by primitive lower rows (c, d) with $5 \mid c$, modulo simultaneous sign. Indeed, these conditions are necessary and sufficient for (c, d) to be the lower row of an element of $\Gamma_0(5)$, and two such matrices with the same lower row differ by left multiplication by an element of Γ_∞. For $\tau = u + iy_0$, the corresponding zero is $z_0 = r^{-1}e^{2\pi i\gamma\tau}$. Since $\operatorname{Im}(\gamma\tau) = y_0/|c\tau + d|^2$, it lies in $\mathbb{D}$ exactly when

$$|c\tau + d| < 1, \tag{38}$$

and its Jensen weight is

$$2\pi y_0 \left(|c\tau + d|^{-2} - 1 \right). \tag{39}$$

Choose the simultaneous sign so that $c \geq 0$. For $c = 0$, primitivity gives $d = 1$; this is the identity coset, whose zero $z_0 = e^{2\pi iu}$ is on the boundary and has Jensen weight zero. For $c > 0$, the inequality $cy_0 < 1$ is necessary. The elementary bounds $2/3 < \log 2 < 1$ and $3 < \pi < 22/7$ give $5 < 1/y_0 < 10$. Since c is a positive multiple of five, the only possible nonidentity value is $c = 5$.

Since $\kappa = 5y_0$, the preceding bounds give $0 < \kappa < 1$. For $c = 5$, we have $|5\tau + d|^2 = (5u + d)^2 + \kappa^2$, so condition (38) becomes $|5u + d| < \nu < 1$. As $0 \leq u < 1$, the only possible integers are $d = 0, -1, \ldots, -5$; primitivity excludes 0 and -5. Thus the complete nonidentity list is $d = -1, -2, -3, -4$. Parameters u for which $\tau = u + iy_0$ represents an elliptic point of $X_0(5)$ form a finite set, hence a null set, and do not affect the integral. Boundary coincidences likewise do not affect it, because their Jensen weight is zero.

For $d = -j$, the interior condition is

$$\frac{j - \nu}{5} < u < \frac{j + \nu}{5}.$$

All four intervals lie in $(0, 1)$. They may overlap, but Jensen's formula sums zeros with multiplicity, so the four coset contributions must be added separately. With $t = 5u - j$, each contribution has Jacobian $du = dt/5$, and $|5\tau + d|^2 = t^2 + \kappa^2$. Integrating (39) therefore gives

$$\begin{aligned} I &= -\log 2 + \frac{8\pi y_0}{5} \int_{-\nu}^{\nu} \left(\frac{1}{t^2 + \kappa^2} - 1 \right) dt \\ &= -\log 2 + \frac{8\pi y_0}{5} \left(\frac{2}{\kappa} \arctan \frac{\nu}{\kappa} - 2\nu \right). \end{aligned}$$

Since $\kappa^2 + \nu^2 = 1$ and $\kappa, \nu > 0$, $\arctan(\nu/\kappa) = \arccos \kappa$. Substitution of $\kappa = 5y_0$ and $2\pi y_0 = \log 2$ now gives

$$I = -\log 2 + \frac{16\pi}{25} \arccos \kappa - \frac{8 \log 2}{5}\nu,$$

which is (36). $\square$

7 Completion of the proof

Lemma 7.1 (rational numerical certificate)**.** *With I as in Lemma 6.1,*

$$I < \frac{183}{500} < \frac{231}{500} < -4\log 2 + 9\log 5 - \frac{45}{4},$$

and

$$-\log 2 + 3\log 5 - \frac{45}{16} > \frac{33}{25}.$$

Proof. We include a certificate involving rational arithmetic only. For $0 < z < 1$, set

$$\mathrm{lg}_N(z) = 2\sum_{k=0}^{N-1} \frac{z^{2k+1}}{2k+1}.$$

Termwise integration of the geometric series gives

$$\mathrm{lg}_N(z) < \log\frac{1+z}{1-z} < \mathrm{lg}_N(z) + \frac{2z^{2N+1}}{(2N+1)(1-z^2)}. \tag{40}$$

We use $z = 1/3, N = 35$ for $\log 2$, and $z = 1/9, N = 18$ for $\log(5/4)$. For $0 < x \le 1$, put

$$\mathrm{at}_K(x) = \sum_{k=0}^{K-1} \frac{(-1)^k x^{2k+1}}{2k+1}.$$

The alternating-series remainder gives, for every positive even K,

$$\mathrm{at}_K(x) < \arctan x < \mathrm{at}_{K+1}(x). \tag{41}$$

We use $K = 60$ at $x = 1/5$ and $K = 24$ at $x = 1/239$ in Machin's identity $\pi = 16\arctan(1/5) - 4\arctan(1/239)$. For $\vartheta = 4\arctan(1/5) - \arctan(1/239)$, rational tangent addition gives $\tan\vartheta = 1$. The inequalities $x - x^3/3 < \arctan x < x$, together with $\pi > 3$, give $0 < \vartheta < \pi/2$, hence $\vartheta = \pi/4$. Finally, we use $K = 200$ to bound the arctangent in

$$\arccos\kappa = \frac{\pi}{2} - \arctan\left(\frac{\kappa}{\sqrt{1-\kappa^2}}\right).$$

This gives the following strict rational intervals.

quantity	strict lower bound	strict upper bound
$\log 2$	$693147180559945/10^{15}$	$693147180559946/10^{15}$
$\log(5/4)$	$223143551314209/10^{15}$	$223143551314210/10^{15}$
π	$3141592653589793/10^{15}$	$3141592653589794/10^{15}$
κ	$551589000381628/10^{15}$	$551589000381629/10^{15}$
$\nu = \sqrt{1-\kappa^2}$	$834116043879983/10^{15}$	$834116043879984/10^{15}$
$\arccos\kappa$	$986528275021999/10^{15}$	$986528275022000/10^{15}$

For clarity, the table records outward decimal roundings, not the intermediate endpoints used recursively. We now specify those exact endpoints. Let $L_- < \log 2 < L_+$ be the two rational bounds from (40) with $(z, N) = (1/3, 35)$, and put

$$\begin{aligned}
\Pi_- &= 16\,\mathrm{at}_{60}(1/5) - 4\,\mathrm{at}_{25}(1/239),\\
\Pi_+ &= 16\,\mathrm{at}_{61}(1/5) - 4\,\mathrm{at}_{24}(1/239),\\
\kappa_- &= \frac{5L_-}{2\Pi_+}, \qquad \kappa_+ = \frac{5L_+}{2\Pi_-}.
\end{aligned}$$

Thus $\Pi_- < \pi < \Pi_+$ and $\kappa_- < \kappa < \kappa_+$. Define

$$\nu_- = 10^{-30}\left\lfloor 10^{30}\sqrt{1-\kappa_+^2}\right\rfloor,$$
$$\nu_+ = 10^{-30}\left(\left\lfloor 10^{30}\sqrt{1-\kappa_-^2}\right\rfloor + 1\right).$$

Integer squaring verifies these floor values and gives $\nu_- < \nu < \nu_+$. Set $t_- = \kappa_-/\nu_+$ and $t_+ = \kappa_+/\nu_-$; exact cross-multiplication gives

$$0 < t_- < \frac{\kappa}{\nu} < t_+ < 1.$$

Finally, define the rational numbers

$$\Theta_- = \frac{\Pi_-}{2} - \mathrm{at}_{201}(t_+), \qquad \Theta_+ = \frac{\Pi_+}{2} - \mathrm{at}_{200}(t_-).$$

Equation (41) gives $\Theta_- < \arccos\kappa < \Theta_+$. Exact cross-multiplication places each of these exact endpoints strictly inside its displayed decimal interval. Thus every entry in the table is certified by rational arithmetic; no rounded endpoint is reused to derive a later row.

Writing $X_{\mathrm{tab},-}$ and $X_{\mathrm{tab},+}$ for the displayed endpoints of a quantity X, put

$$U_{\mathrm{tab}} = -(\log 2) * \mathrm{tab}, - + \frac{16}{25}\pi * \mathrm{tab}, + (\arccos\kappa) * \mathrm{tab}, +$$
$$-\frac{8}{5}(\log 2) * \mathrm{tab}, - \nu_{\mathrm{tab},-}.$$

The signs in (36) now give

$$I < U_{\mathrm{tab}} = \frac{45665164124379924046682621253}{125000000000000000000000000000} < \frac{183}{500}.$$

Since $\log 5 = 2\log 2 + \log(5/4)$, the two remaining expressions are

$$-4\log 2 + 9\log 5 - \frac{45}{4} = 14\log 2 + 9\log(5/4) - \frac{45}{4},$$
$$-\log 2 + 3\log 5 - \frac{45}{16} = 5\log 2 + 3\log(5/4) - \frac{45}{16}.$$

Their lower endpoints give, respectively,

$$-4\log 2 + 9\log 5 - \frac{45}{4} > \frac{462352489667111}{10^{15}} > \frac{231}{500}$$

and

$$-\log 2 + 3\log 5 - \frac{45}{16}\frac{82666659796397}{62500000000000} > \frac{33}{25}.$$

□

Proof of Theorem 1.1. Assume for contradiction that $\zeta_5(3) \in \mathbf{Q}$. Then $\xi \in \mathbf{Q}$, and all four germs

$$F_1 = 1, \qquad F_2 = H, \qquad F_3 = \delta H, \qquad F_4 = \delta^2 H$$

belong to $\mathbf{Q}[[f]]$. Equation (26) expresses $\delta^3 H$ as a $\mathbf{Q}(f)$-linear combination of $1, H, \delta H$. Hence the $\mathbf{Q}(f)$-span of $1, H, \delta H, \delta^2 H$ is stable under δ, so each of its elements is holonomic. Proposition 4.3 says that the four displayed germs are linearly independent over $\mathbf{Q}(f)$.

We next verify the denominator hypothesis of Theorem 3.1. Write $\xi = a_0/b_0$ in lowest terms with $b_0 > 0$. Since $D_n = \Lambda(n)$, Lemma 5.1 shows that the common multiplier $C_0 = b_0$ in (7) works with the capping matrix

$$\mathbf{b} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{42}$$

Indeed, the first row requires no denominator, while each coefficient in the last three rows becomes integral after multiplication by $b_0 D_n^3$. Thus $m = 4$, and each column has $u_j = b_j = 1$. Therefore

$$\tau(\mathbf{b}) = 3 - \frac{3}{16} = \frac{45}{16}. \tag{43}$$

It remains to check the local hypotheses and compute the two adelic quantities. The maps in (35) are analytic on neighborhoods of the corresponding closed unit disks, vanish simply at the origin, and are the identity at all primes other than five. At five, $\varphi_5(\{|z| * 5 < 1\})$ lies in the continuation disk of Lemma 5.2; at every $\ell \neq 5$, the coefficient estimate following that lemma gives analytic continuation on the open unit disk. At the complex place, the last paragraph of Section 5 gives meromorphic continuation of the pullbacks by $\varphi * \infty$ to a neighborhood of the closed unit disk, with at most two poles. Thus all local hypotheses of Theorem 3.1 hold for the four germs.

For the maps in (35), normalized by $|5| * 5 = 5^{-1}$,

$$\mathcal{A} = I + 3\log 5, \tag{44}$$

$$\mathcal{L} = -\log 2 + 3\log 5. \tag{45}$$

Indeed, $\mathfrak{f}'(0) = 1$, so $\varphi * \infty'(0) = 1/2$. At five, both the derivative and the boundary supremum contribute $\log|5^{-3}|_5 = 3\log 5$; all other finite places contribute zero. Put

$$\mathcal{D} = \mathcal{L} - \tau(\mathbf{b}) = -\log 2 + 3\log 5 - \frac{45}{16}.$$

Lemma 7.1 gives $\mathcal{D} > 33/25 > 0$. Theorem 3.1 therefore gives

$$4 \leq \frac{I + 3\log 5}{-\log 2 + 3\log 5 - 45/16}. \tag{46}$$

The first chain in Lemma 7.1 gives $I < -4\log 2 + 9\log 5 - 45/4$. Equivalently,

$$4\mathcal{D} - (I + 3\log 5) = -4\log 2 + 9\log 5 - \frac{45}{4} - I > 0.$$

Since $\mathcal{D} > 0$, this proves

$$\frac{I + 3\log 5}{-\log 2 + 3\log 5 - 45/16} < 4. \tag{47}$$

Equations (46) and (47) contradict each other. The rationality assumption was therefore false. Since $\zeta_5(3) \in \mathbf{Q}_5$ by construction of the Kubota–Leopoldt value, Theorem 1.1 follows; the nonvanishing is the special case $\zeta_5(3) \neq 0 \in \mathbf{Q}$. $\square$

Comparison with the elementary level-five estimate

Calegari's elementary level-five approximants have exponent

$$\frac{3}{3/2+3/\log 5} = 0.891794\ldots < 1,$$

and therefore do not prove Theorem 1.1 [5, §3.2]. The argument above uses both the finite-place holonomy contribution and the optimized row-wise invariant 45/16. If that invariant is replaced by the unoptimized value 3, the resulting quotient is 4.575…, which yields no contradiction.

Remark 7.2 (the margin). The quotient in (46) equals 3.9266…, which is less than two percent below the critical value 4. Two comments on the robustness of this margin are in order; neither is used in the proof.

First, the only freedom in the archimedean data is the template φ_∞. Within the family $\varphi_\infty(z) = \mathfrak{f}(rz)$ of disk templates, numerical evaluation of the energy gives the quotients 3.957, 3.9266, 3.9215, and 3.961 for $r = 0.45,\ 0.5,\ 0.55,\ 0.6$, respectively. The choice $r = 1/2$, for which Lemma 6.1 is particularly simple because only the cosets with $c = 5$ contribute, is close to the numerical minimum 3.916961…, attained at $r \approx 0.534813$. A noncircular template, such as the lune-shaped templates used in [7, §5.2], is a possible route to a better bound, but no improvement for the present level-five construction is established here.

Second, the coefficient valuations can be checked independently of Lemma 5.2. Let B_k be the Bernoulli numbers defined by $t/(e^t-1) = \sum_{k\geq 0} B_k t^k/k!$, so $B_1 = -1/2$. The residue-class expansion of the Volkenborn integral for $\zeta_5(3) = L_5(3, \omega_5^2)$ gives

$$\xi = \frac{1}{20}\sum_{a=1}^{4}\sum_{k\geq 0}\frac{(-1)^k(k+1)5^k B_k}{a^{k+2}};$$

see [9, §2.3] and [16, Chapter 5]. Since $v_5(B_k) \geq -1$, truncating after $k = 150$ determines ξ modulo 5^{149}. Using this value in $h_n = \alpha_n + \xi\beta_n$, with the notation of Lemma 5.1, gives

$$3n - 9 \leq v_5(h_n) \leq 3n - 3 \qquad (1 \leq n \leq 40).$$

For example, the first eight valuations are $0, 3, 5, 8, 10, 13, 15, 18$. These are finite checks, not a proof of a convergence radius or of divergence on its boundary. They do not uniquely determine the constant term: replacing ξ by $\xi + 5^{149}$ leaves all forty valuations unchanged, since $\beta_n \in \mathbf{Z}$ and all the tested valuations are below 149.

Role of computation and reproducibility

The public repository `github.com/shivakintali/zeta-5-3` contains two initial verification scripts. The script `verifylevel5ode.py` checks formal inversion and the equations for A and AG through degree 80. The script `energy_exact.py` proves the final energy, threshold, and positive-denominator comparisons using rational intervals.

The accompanying `verification/` directory in the same repository, provides three further checks, using only the Python standard library. The script `checkodelocal.py` verifies the rational identities and indicial polynomial exactly, and independently checks the Fourier identities through degree 39 and the denominator rows through degree 38. The script `energyindependentcheck.py` verifies the coset-enumeration bounds, every displayed interval endpoint, and the rational comparisons in Lemma 7.1; it also reports separately the exploratory numerical optimizations. The script `checkpadicclaim.py` reproduces the forty coefficient valuations in Remark 7.2 from the Bernoulli expansion above. The accompanying README records how to run the checks and distinguishes exact assertions from floating-point estimates.

The modular and denominator arguments prove identities and divisibility in all degrees; the finite series checks corroborate those proofs. The rational certificate is part of the proof of the final strict inequality. The numerical optimizations and finite 5-adic experiment are not used in the proof of Theorem 1.1.

8 Further work

In shis section we examine the prospects for extending the method to $\zeta_p(3)$ at other primes. We first quantify the obstruction at $p = 7$, then discuss the denominator, analytic, and geometric barriers at general primes. We identify improvements in denominator estimates, analytic domains, and auxiliary constructions that could overcome these barriers, together with the independence and continuation results needed for an irrationality proof.

The level-seven analogue and its quantitative obstruction

The modular construction has a natural analogue at $p = 7$, but the same denominator estimates and disk templates do not give an irrationality proof there. We describe this limitation explicitly. With $q = e^{2\pi i\tau}$ and $\theta = q\,d/dq$, put

$$f_7(\tau) = \left(\frac{\eta(7\tau)}{\eta(\tau)}\right)^4 = q\prod_{n\geq 1}(1+q^n+\cdots+q^{6n})^4,$$
$$A_7(\tau) = \frac{\theta f_7}{f_7} = \frac{7E_2(7\tau)-E_2(\tau)}{6} = 1+4\sum_{n\geq 1}\sum_{\substack{d\mid n\\ 7\nmid d}} d\,q^n,$$
$$G_7(q) = \sum_{n\geq 1}\left(\sum_{\substack{d\mid n\\ 7\nmid d}} d^{-3}\right)q^n.$$

The eta quotient f_7 has a simple zero at the cusp ∞, a simple pole at the cusp 0, and no other zeros or poles. Thus it is a degree-one coordinate on the genus-zero curve $X_0(7)$. Write $x = f_7$, let $q_7(x) \in x + x^2\mathbf{Z}[[x]]$ be the inverse formal series, and define

$$H_7(x) = A_7(q_7(x))\big(G_7(q_7(x)) + \zeta_7(3)/2\big), \qquad \delta_7 = x\frac{d}{dx}.$$

Under the assumption $\zeta_7(3)/2 = a/b \in \mathbf{Q}$, with $b > 0$, the integrality of A_7 and q_7 gives exactly the same coefficient estimate as Lemma 5.1:

$$bD_n^3[x^n]\delta_7^j H_7 \in \mathbf{Z} \qquad (j = 0,1,2), \qquad D_n = \operatorname{lcm}(1,\dots,n), \quad D_0 = 1.$$

Even if one establishes the level-seven analogue of the exact-order result in Proposition 4.3, the four germs $1, H_7, \delta_7 H_7, \delta_7^2 H_7$ would therefore have the same denominator capping matrix (42) and the same invariant $\tau(\mathbf{b}) = 45/16$. We do not need that exact-order assertion for the obstruction below: it shows that these numerical inputs would be insufficient even with four independent germs.

The crucial change is in the finite-place contribution. The Fricke involution $w_7\tau = -1/(7\tau)$ satisfies

$$f_7(w_7\tau) = \frac{1}{49 f_7(\tau)}.$$

In this normalization, the two ordinary components and the intervening supersingular annulus are described by

$$|x|_7 \le 1, \qquad |x|_7 \ge 49, \qquad 1 < |x|_7 < 49,$$

respectively. The annulus corresponds to the supersingular invariant $j = 1728 \bmod 7$; see [15, §3.4], whose parameter is $T = 1/f_7$. The Eisenstein series $G_7 + \zeta_7(3)/2$ has U_7-eigenvalue one. Hence the same use of Buzzard's continuation theorem as in Lemma 5.2 supplies analyticity of H_7 and its Euler derivatives on the strict disk $|x|_7 < 49$. The local map $z \mapsto 7^{-2}z$ consequently contributes $B_7 := 2\log 7$, which is smaller than the level-five contribution $3\log 5$. No larger continuation disk is supplied by this argument.

To examine the complex input, denote the q-expansion of f_7 by $\mathfrak{f}_7(q)$, and use the maps $\varphi_\infty(z) = \mathfrak{f}_7(e^{-s}z)$, $s > 0$. Define

$$I_7(s) = \iint_{\mathbf{T}^2} \log|\mathfrak{f}_7(e^{-s}z) - \mathfrak{f}_7(e^{-s}w)|\, d\mu(z)\, d\mu(w).$$

The Jensen argument of Lemma 6.1 gives the finite sum

$$I_7(s) = -s + \sum_{\substack{c\ge 1,\ 7|c \\ cs<2\pi}} \frac{4\pi\phi(c)}{c^2}\left(\arccos a_c - a_c\sqrt{1-a_c^2}\right), \qquad a_c = \frac{cs}{2\pi}, \tag{48}$$

where $\phi(c)$ is Euler's totient function. Indeed, the nonidentity cosets are indexed by primitive lower rows (c, d), with $7 \mid c$ and $c > 0$. For $\tau = u + is/(2\pi)$, the interior-preimage condition is $|c\tau + d| < 1$. For each admissible c, exactly $\phi(c)$ intervals occur as u ranges from zero to one. Integrating their Jensen weights gives the summand in (48); all summands are nonnegative. Elliptic coincidences form a null set and do not change the integral.

At the radius $e^{-s} = 1/2$, only $c = 7$ contributes, so

$$I_7(\log 2) = -\log 2 + \frac{24\pi}{49}\arccos a - \frac{12\log 2}{7}\sqrt{1-a^2}, \qquad a = \frac{7\log 2}{2\pi}.$$

The numerical comparison with the level-five inputs is

Quantity at complex radius 1/2	Level five	Level seven
Finite-place contribution	4.828314	3.891820
Complex logarithmic energy	0.365321	−0.388741
Holonomy quotient	3.926640	9.071268

Thus, granting the four-function independence, the direct level-seven application would give only

$$4 \le \frac{I_7(\log 2) + 2\log 7}{-\log 2 + 2\log 7 - 45/16} = 9.071267\ldots,$$

which provides no contradiction.

In fact, changing the radius within this family cannot make the quotient less than four. Put

$$Q_7(s) = \frac{I_7(s) + B_7}{B_7 - s - 45/16}, \qquad 0 < s < B_7 - 45/16.$$

For $0 < s < 2\pi/7$, retain just the $c = 7$ summand in (48), and set

$$E_7(s) = \frac{24\pi}{49}\left(\arccos a - a\sqrt{1-a^2}\right), \qquad a = \frac{7s}{2\pi}.$$

Set $E_7(s) = 0$ for $s \geq 2\pi/7$. Then $I_7(s) \geq -s + E_7(s)$, and the margin needed for a contradiction satisfies

$$4(B_7 - s - 45/16) - (I_7(s) + B_7) \leq 6\log 7 - \frac{45}{4} - 3s - E_7(s).$$

On $0 < s < 2\pi/7$, the derivative of the right side is $-3 + (24/7)\sqrt{1-(7s/(2\pi))^2}$; beyond that interval it is -3. Its global maximum is therefore attained at $s_* = \pi\sqrt{15}/28$, and equals

$$6\log 7 - \frac{45}{4} - \frac{24\pi}{49}\left(\arccos\frac{\sqrt{15}}{8} + \frac{7\sqrt{15}}{64}\right) = -1.865788\ldots < 0. \tag{49}$$

The sign does not depend on numerical rounding: since $\log 7 < 2$, $\sqrt{15}/8 < 1/2$, and $\pi > 3$, the expression is less than $3/4 - 8\pi^2/49 < 0$. Consequently $Q_7(s) > 4$ for every radius with positive denominator. This excludes the entire family $\mathfrak{f}_7(rz)$, $0 < r < 1$, with these finite-place data and the invariant $45/16$, not arbitrary complex templates or auxiliary constructions.

For scale, numerical minimization of (48) gives

$$\min_{0<s<B_7-45/16} Q_7(s) \approx 6.570557, \qquad s \approx 0.27260216, \qquad r = e^{-s} \approx 0.76139564.$$

These decimals are exploratory numerical estimates; the failure of the radius family follows from the analytic bound (49). Keeping the same family and the count four, but replacing $45/16$ by an improved effective denominator invariant t, a contradiction would require

$$t < \frac{3B_7 - 4s - I_7(s)}{4}$$

for some $s > 0$. Numerical maximization of this threshold gives approximately 2.345998610, at $r \approx 0.63844012$, compared with the present value $45/16 = 2.8125$. The maximizing radius for this threshold differs from the minimizing radius for Q_7. Thus possible extensions would need stronger denominator control, a complex template outside the family $\mathfrak{f}_7(rz)$, or a different auxiliary construction with a better balance of independence, arithmetic, and continuation.

Extension to general primes: barriers and required improvements

The Eisenstein-series framework is available for every prime, but its existence alone does not give an irrationality proof. For a prime p, define

$$G_p(q) = \sum_{n\geq 1}\left(\sum_{\substack{d|n\\ p\nmid d}} d^{-3}\right) q^n,$$

$$A_p(\tau) = \frac{pE_2(p\tau) - E_2(\tau)}{p-1} = 1 + \frac{24}{p-1}\sum_{n\geq 1}\sum_{\substack{d|n\\ p\nmid d}} d\, q^n,$$

$$\mathcal{H}_p(q) = A_p(\tau)\big(G_p(q) + \zeta_p(3)/2\big).$$

The factor $G_p + \zeta_p(3)/2$ is the weight--2 overconvergent Eisenstein series with U_p-eigenvalue one [5, §2.2]; thus $\mathcal{H}_p$ is a weight-zero overconvergent modular function. Under the assumption $\zeta_p(3)/2 = a/b \in \mathbf{Q}$, the coefficient estimate

$$b(p-1)D_n^3[q^n]\mathcal{H}_p \in \mathbf{Z}$$

follows directly from the displayed expansions. The multiplier $b(p-1)$ is independent of n. Passing from this q-expansion to an algebraic coordinate suitable for holonomy, however, requires its own denominator estimate; the q-estimate does not by itself verify the hypotheses of Theorem 3.1 in a different coordinate.

The quantitative target is the same for any proposed application of that theorem. If m independent germs have effective denominator invariant $t = \tau(\mathbf{b})$ and common admissible local maps, a contradiction requires

$$\mathcal{L} > t, \qquad m(\mathcal{L} - t) - \mathcal{A} > 0. \tag{50}$$

The first inequality makes the holonomy bound applicable; the second makes its upper bound smaller than the number of independent germs. All three quantities m, t, and the analytic data must belong to the same construction. In particular, the number $m = 4$ and the invariant $45/16$ cannot be retained without verification after changing the coordinate, the differential system, or the auxiliary functions.

For prime level, $X_0(p)$ has genus zero precisely when $p \in \{2, 3, 5, 7, 13\}$. In these cases the normalized Hauptmodul and its Fricke transform are

$$f_p(\tau) = \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^{24/(p-1)}, \qquad f_p(w_p\tau) = \frac{p^{-12/(p-1)}}{f_p(\tau)}, \qquad w_p\tau = -\frac{1}{p\tau}.$$

The integral product expansion and its inverse preserve the D_n^3 denominator bound. The ordinary component containing ∞ is $|f_p|_p \leq 1$ [13, Theorem 1]; the Fricke involution identifies the other component as $|f_p|_p \geq p^{12/(p-1)}$. Consequently, Buzzard's continuation theorem gives the strict disk with radius $R_p = p^{12/(p-1)}$ for the weight-zero product, just as at levels five and seven. The corresponding logarithmic gain is $B_p = \log R_p$:

Prime p	Continuation radius R_p	Gain B_p
2	2^{12}	8.317766
3	3^6	6.591674
5	5^3	4.828314
7	7^2	3.891820
13	13	2.564949

This formula is being used only for these genus-zero coordinates; it is not an assertion of a coordinate disk of that radius at arbitrary prime level. The gain decreases across the table, whereas the denominator invariant of the proposed four-germ construction remains 45/16.

Level thirteen already exhibits a stronger positivity obstruction than level seven. For the radial template $\varphi_\infty(z) = \mathfrak{f}_{13}(rz)$, $0 < r < 1$, and the standard finite-place maps, one has

$$\mathcal{L} = \log r + \log 13 < \log 13 < \frac{45}{16}.$$

Thus the first inequality in (50) fails for every such radius. In fact, this particular obstruction persists for any template $\mathfrak{f}_{13} \circ \psi$ with $\psi : \mathbb{D} \to \mathbb{D}$ holomorphic and $\psi(0) = 0$: Schwarz's lemma gives $|\psi'(0)| \leq 1$, and $\mathfrak{f}_{13}'(0) = 1$. Here the finite-place maps and denominator invariant are held fixed.

The size of a possible arithmetic improvement can also be estimated. For $p = 7, 13$, let $I_p(s)$ be the energy of $\mathfrak{f}_p(e^{-s}z)$. Formula (48) holds with $7 \mid c$ replaced by $p \mid c$, by the same degree-one coordinate and coset argument. With four independent germs, a replacement invariant t must satisfy

$$t < \sup_{s>0} \frac{3B_p - 4s - I_p(s)}{4}.$$

Numerical maximization gives the following approximate thresholds:

Prime p	Threshold for t	Maximizing radius e^{-s}
7	2.345998610	0.63844012
13	1.606485412	0.75445629

These are exploratory feasibility estimates for the specified radial family, not denominator bounds that have been proved. They quantify how far its present cost $45/16 = 2.8125$ is from a useful arithmetic estimate. Any successful application must also verify the positive denominator condition in (50).

For primes outside $\{2, 3, 5, 7, 13\}$, there is an additional geometric problem: $X_0(p)$ has positive genus, already at $p = 11$. There is no degree-one Hauptmodul, and the continuation domain supplied on the modular curve need not become a single disk in a chosen local parameter. One possible approach is to choose a finite map $X_0(p) \to \mathbf{P}^1_x$ and express the differential system over $\mathbf{Q}(x)$, starting from the function field $\mathbf{Q}(X_0(p))$. This can increase its rank and introduces degree, ramification, and denominator costs that must be tracked explicitly. Another approach is to work directly on $X_0(p)$ using a suitable arithmetic holonomy bound on a curve. Either approach must supply the actual local continuation maps or their geometric counterparts and the associated complex energy and finite-place estimates. The genus-zero Jensen formula cannot simply be reused for a map of higher degree without accounting for its fibers.

Functional independence is a separate requirement. The symmetric-square monodromy argument of Proposition 4.3 suggests a route to an exact-order statement over the relevant function field, but a generalization must still exclude lower-order inhomogeneous relations. Merely deriving an equation of order three proves an upper bound on the order. Likewise, once a differential system closes, adjoining higher derivatives does not automatically increase the number of independent germs. New auxiliary functions must have verified independence, controlled denominators, and the required common continuation domains; integrality of their coefficients alone supplies none of the additional continuation.

Thus a generalization requires a better quantitative balance among the number of independent germs, their denominator bounds, and the analytic data, together with a treatment of the geometry where necessary. Sharper arithmetic estimates would have to establish prime-by-prime denominator savings that lower the effective invariant beyond the existing D_n^3 estimate. Better complex templates would have to improve the balance of the derivative at the origin and the boundary energy, with a rigorous bound for that energy. Noncircular templates have a precedent in the proof of irrationality of $\zeta_2(5)$ in [7, §5.2]; their sufficiency for $\zeta_7(3)$ is not established here.

Within the stated holonomy bound, a nonlinear finite-place map whose image remains in the known continuation disk cannot improve on the radial map. Indeed, its derivative and Gauss norm satisfy

$$\log|\varphi_p'(0)| * p \le \sup *|z|_p = 1 \log|\varphi_p(z)|_p \le \log R_p,$$

so its contribution to $m\mathcal{L} - \mathcal{A}$, for $m > 1$, is at most $(m-1)\log R_p$, already attained by the radial map. An improvement at the finite places would therefore require a larger continuation region, different auxiliary functions, or a stronger bound that uses additional geometric information about the domain.

An enlarged auxiliary system would need to gain enough independence to compensate for all of its arithmetic and analytic costs. At positive genus, these estimates must be combined with a fully quantified finite-map or curve-based formulation.

Level seven is therefore a useful next test: it isolates the quantitative obstruction without introducing positive genus. A concrete advance would be admissible data for four independent germs satisfying $4(\mathcal{L}-t)-\mathcal{A}>0$ with $\mathcal{L}>t$, or a new system satisfying (50) with its actual rank. A result for every prime would also prove $\zeta_p(3)\neq 0$ for every prime, the case $n=1$ of [5, Conjecture 2.5]; the general nonvanishing problem is emphasized again in [7, §5.2]. The constructions and estimates in the present paper do not resolve that problem uniformly in p.

Use of AI: We prepared a list of proof strategies and techniques used in the irrationality proofs of zeta values and p-adic zeta values. We fed this to multiple AI tools (Claude, ChatGPT and two open-weight models with open-source harnesses) and prompted these AI tools to prove irrationality of $\zeta(n)$ and $\zeta_p(n)$ for several values of n,p and also subsets of zeta values (to prove that at least one of the values in the subset is irrational). When an AI tool was hallucinating, stuck, or going in the wrong direction, we stopped that AI tool, corrected the intermediate `.md` files, added more specific new ideas and switched to a different AI tool with more detailed fresh prompts. Equations written by one AI tool were critiqued, verified and expanded by a completely different AI tool in an iterative manner, till we were satisfied with their correctness and details. Irrationality of $\zeta_5(3)$ is the first result we obtained in this domain. The certificates code was written by Codex and verified by the author. Section 8 was written by the author and then verified and expanded using AI. The author is solely responsible for the correctness of the proofs.

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